

UNITARY INVARIANTS ON THE UNIT BALL OF $B(\mathcal{H})^n$

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ABSTRACT. In this paper, we introduce a unitary invariant

$$\Gamma : [B(\mathcal{H})^n]_1^- \rightarrow \mathbb{N}_\infty \times \mathbb{N}_\infty \times \mathbb{N}_\infty, \quad \mathbb{N}_\infty := \mathbb{N} \cup \{\infty\},$$

defined in terms of the characteristic function Θ_T , the noncommutative Poisson kernel K_T , and the defect operator Δ_T associated with $T \in [B(\mathcal{H})^n]_1^-$. We show that the map Γ detects the pure row isometries in the closed unit ball of $B(\mathcal{H})^n$ and completely classify them up to a unitary equivalence. We also show that Γ detects the pure row contractions with polynomial characteristic functions and completely non-coisometric row contractions, while the pair (Γ, Θ_T) is a complete unitary invariant for these classes of row contractions.

The unitary invariant Γ is extracted from the theory of characteristic functions and noncommutative Poisson transforms, and from the geometric structure of row contractions with polynomial characteristic functions which are studied in this paper. As an application, we characterize the row contractions with constant characteristic function. In particular, we show that any completely non-coisometric row contraction T with constant characteristic function is homogeneous, i.e., T is unitarily equivalent to $\varphi(T)$ for any free holomorphic automorphism φ of the unit ball of $B(\mathcal{H})^n$.

Under a natural topology, we prove that the free holomorphic automorphism group $\text{Aut}(B(\mathcal{H})_1^n)$ is a metrizable, σ -compact, locally compact group, and provide a concrete unitary projective representation of it in terms of noncommutative Poisson kernels.

INTRODUCTION

An n -tuple $T = (T_1, \dots, T_n)$ of bounded linear operators is called row contraction if it belongs to the closed unit ball

$$[B(\mathcal{H})^n]_1^- := \{(X_1, \dots, X_n) \in B(\mathcal{H})^n : X_1 X_1^* + \dots + X_n X_n^* \leq I\},$$

where $B(\mathcal{H})$ is the algebra of bounded linear operators on a Hilbert space \mathcal{H} . In recent years, there has been exciting progress in multivariable operator theory on $[B(\mathcal{H})^n]_1^-$, especially in connection with dilation theory and unitary invariants for n -tuples of operators such as characteristic function, curvature and Euler characteristic, entropy, joint numerical radius and joint ρ -operator radius (see [5], [11], [12], [15], [16] and the references therein).

A central problem in multivariable operator theory is the classification, up to a unitary equivalence, of n -tuples of operators. In this paper, we introduce a new unitary invariant

$$\Gamma : [B(\mathcal{H})^n]_1^- \rightarrow \mathbb{N}_\infty \times \mathbb{N}_\infty \times \mathbb{N}_\infty$$

which is extracted from the geometric structure of row contractions with polynomial characteristic functions and the theory of noncommutative Poisson transforms on the unit ball of $B(\mathcal{H})^n$. We use Γ to detect and classify certain classes of n -tuples of operators in the unit ball of $B(\mathcal{H})^n$.

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In Section 1, we show that a row contraction $T = (T_1, \dots, T_n)$ has polynomial characteristic function of degree $m \in \mathbb{N} := \{0, 1, \dots\}$ if and only if T_i admits a canonical upper triangular representation

$$T_i = \begin{bmatrix} V_i & * & * \\ 0 & N_i & * \\ 0 & 0 & W_i \end{bmatrix}, \quad i = 1, \dots, n,$$

where (V_1, \dots, V_n) is a pure isometry, (N_1, \dots, N_n) is a nilpotent row contraction of order m , and (W_1, \dots, W_n) is a coisometry. In the particular case when $n = 1$ and T is a completely non-unitary (c.n.u.) contraction, we recover a recent result of Foiaş and Sarkar [3]. The results of Section 1 lead to the definition of the map

$$\Gamma : [B(\mathcal{H})^n]_1^- \rightarrow \mathbb{N}_\infty \times \mathbb{N}_\infty \times \mathbb{N}_\infty, \quad \Gamma(T) := (p, m, q)$$

by setting $m := \deg(\Theta_T)$, $q := \dim(\ker K_T)$, and

$$p := \begin{cases} \dim(\mathcal{D}_m \ominus \mathcal{D}_{m+1}) & \text{if } m \in \mathbb{N} \\ \dim \overline{\Delta_T \mathcal{H}} & \text{if } m = \infty, \end{cases}$$

where $\mathcal{D}_m := \overline{\text{span}}\{T_\beta \Delta_T h : h \in \mathcal{H}, |\beta| \geq m\}$, Θ_T is the characteristic function, K_T is the noncommutative Poisson kernel, and Δ_T is the defect operator associated with $T \in [B(\mathcal{H})^n]_1^-$.

In Section 2, we show that the map Γ detects the pure row isometries in the closed unit ball of $B(\mathcal{H})^n$ and completely classify them up to a unitary equivalence. We also show that Γ detects the pure row contractions with polynomial characteristic functions and completely non-coisometric (c.n.c.) row contractions, while the pair (Γ, Θ_T) is a complete unitary invariant for these classes of row contractions. As an application of the results from Section 1, we prove that the characteristic function Θ_T is a constant if and only if T admits a canonical upper triangular representation

$$T_i = \begin{bmatrix} V_i & * \\ 0 & W_i \end{bmatrix}, \quad i = 1, \dots, n,$$

where $V := (V_1, \dots, V_n)$ is a pure isometry and $W := (W_1, \dots, W_n)$ is a coisometry.

In Section 3, we prove that a c.n.c row contraction T is homogeneous if and only if $\Theta_T \circ \Psi^{-1}$ coincides with the characteristic function Θ_T for any Ψ in the group $\text{Aut}(B(\mathcal{H})_1^n)$ of free holomorphic automorphisms of $[B(\mathcal{H})^n]_1$. In particular, we show that any c.n.c row contraction T with constant characteristic function is homogeneous, i.e., T is unitarily equivalent to $\varphi(T)$ for any $\varphi \in \text{Aut}(B(\mathcal{H})_1^n)$. Moreover, we show that

$$\varphi_i(T) = U_\varphi T_i U_\varphi^*, \quad i = 1, \dots, n,$$

where U_φ is a unitary operator satisfying relation $U_\varphi U_\psi = c(\varphi, \psi) U_{\varphi \circ \psi}$ for some complex number $c(\varphi, \psi) \in \mathbb{T}$. We remark that in the single variable case ($n = 1$) we find again some of the results obtain by Clark, Misra, and Bagchi (see [1], [2]).

The theory of characteristic functions for row contractions [5] was used in [16] to determine the group $\text{Aut}(B(\mathcal{H})_1^n)$ of all free holomorphic automorphisms of $[B(\mathcal{H})^n]_1$. We obtained a characterization of the unitarily implemented automorphisms of the Cuntz-Toeplitz algebra $C^*(S_1, \dots, S_n)$, which leave invariant the noncommutative disc algebra \mathcal{A}_n , in terms of noncommutative Poisson transforms. This result provided new insight into Voiculescu's group [18] of automorphisms of the Cuntz-Toeplitz algebra and revealed new connections with noncommutative multivariable operator theory. Employing some techniques from [16], we prove that, with respect to the metric

$$d_{\mathcal{E}}(\phi, \psi) := \|\phi - \psi\|_\infty + \|\phi^{-1}(0) - \psi^{-1}(0)\|, \quad \phi, \psi \in \text{Aut}(B(\mathcal{H})_1^n),$$

the free holomorphic automorphism group $\text{Aut}(B(\mathcal{H})_1^n)$ is a σ -compact, locally compact group, and we provide a concrete unitary projective representation of it in terms of noncommutative Poisson kernels.

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1. ROW CONTRACTIONS WITH POLYNOMIAL CHARACTERISTIC FUNCTIONS

Let H_n be an n -dimensional complex Hilbert space with orthonormal basis e_1, e_2, \dots, e_n , where $n = 1, 2, \dots$. We consider the full Fock space of H_n defined by

$$F^2(H_n) := \mathbb{C}1 \oplus \bigoplus_{k \geq 1} H_n^{\otimes k},$$

where $H_n^{\otimes k}$ is the (Hilbert) tensor product of k copies of H_n . Define the left (resp. right) creation operators S_i (resp. R_i), $i = 1, \dots, n$, acting on $F^2(H_n)$ by setting

$$S_i \varphi := e_i \otimes \varphi, \quad \varphi \in F^2(H_n),$$

(resp. $R_i \varphi := \varphi \otimes e_i$, $\varphi \in F^2(H_n)$). The noncommutative disc algebra \mathcal{A}_n (resp. \mathcal{R}_n) is the norm closed algebra generated by the left (resp. right) creation operators and the identity. The noncommutative analytic Toeplitz algebra F_n^∞ (resp. \mathcal{R}_n^∞) is the weakly closed version of \mathcal{A}_n (resp. \mathcal{R}_n). These algebras were introduced (see [6], [7], [9]) in connection with a noncommutative von Neumann type inequality [19].

Let \mathbb{F}_n^+ be the unital free semigroup on n generators g_1, \dots, g_n and the identity g_0 . The length of $\alpha \in \mathbb{F}_n^+$ is defined by $|\alpha| := 0$ if $\alpha = g_0$ and $|\alpha| := k$ if $\alpha = g_{i_1} \cdots g_{i_k}$, where $i_1, \dots, i_k \in \{1, \dots, n\}$. If $(X_1, \dots, X_n) \in B(\mathcal{H})^n$, where $B(\mathcal{H})$ is the algebra of all bounded linear operators on the Hilbert space \mathcal{H} , we set $X_\alpha := X_{i_1} \cdots X_{i_k}$ and $X_{g_0} := I_{\mathcal{H}}$. We denote $e_\alpha := e_{i_1} \otimes \cdots \otimes e_{i_k}$ and $e_{g_0} := 1$.

We recall ([5], [8]) a few facts concerning multi-analytic operators on Fock spaces. We say that a bounded linear operator M acting from $F^2(H_n) \otimes \mathcal{K}$ to $F^2(H_n) \otimes \mathcal{K}'$ is multi-analytic with respect to S_1, \dots, S_n if

$$M(S_i \otimes I_{\mathcal{K}}) = (S_i \otimes I_{\mathcal{K}'})M, \quad i = 1, \dots, n.$$

We can associate with M a unique formal Fourier expansion

$$M(R_1, \dots, R_n) := \sum_{\alpha \in \mathbb{F}_n^+} R_\alpha \otimes \theta_{(\alpha)},$$

where $\theta_{(\alpha)} \in B(\mathcal{K}, \mathcal{K}')$. We know that

$$M = \text{SOT-} \lim_{r \rightarrow 1} \sum_{k=0}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} R_\alpha \otimes \theta_{(\alpha)},$$

where, for each $r \in [0, 1)$, the series converges in the operator norm. Moreover, the set of all multi-analytic operators in $B(F^2(H_n) \otimes \mathcal{K}, F^2(H_n) \otimes \mathcal{K}')$ coincides with $R_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$, the WOT-closed operator space generated by the spatial tensor product. A multi-analytic operator is called inner if it is an isometry. We remark that similar results are valid for multi-analytic operators with respect to the right creation operators R_1, \dots, R_n .

According to [14], a map $F : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes}_{\min} B(\mathcal{E}, \mathcal{G})$ is called *free holomorphic function* on $[B(\mathcal{H})^n]_\gamma$, $\gamma > 0$, with coefficients in $B(\mathcal{E}, \mathcal{G})$ if there exist $A_{(\alpha)} \in B(\mathcal{E}, \mathcal{G})$, $\alpha \in \mathbb{F}_n^+$, such that

$$F(X_1, \dots, X_n) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} X_\alpha \otimes A_{(\alpha)},$$

where the series converges in the operator norm topology for any $(X_1, \dots, X_n) \in [B(\mathcal{H})^n]_\gamma$, where

$$[B(\mathcal{H})^n]_\gamma := \{(X_1, \dots, X_n) \in B(\mathcal{H})^n : \|X_1 X_1^* + \cdots + X_n X_n^*\|^{1/2} < \gamma\},$$

For simplicity, throughout this paper, $[X_1, \dots, X_n]$ denotes either the n -tuple $(X_1, \dots, X_n) \in B(\mathcal{H})^n$ or the operator row matrix $[X_1 \cdots X_n]$ acting from $\mathcal{H}^{(n)}$, the direct sum of n copies of a Hilbert space \mathcal{H} , to \mathcal{H} . The *characteristic function* associated with an arbitrary row contraction $T := [T_1, \dots, T_n]$, $T_i \in B(\mathcal{H})$, was introduced in [5] (see [17] for the classical case $n = 1$) and it was proved to be a complete unitary invariant for completely non-coisometric row contractions. The characteristic function of T is a multi-analytic operator with respect to S_1, \dots, S_n ,

$$\tilde{\Theta}_T : F^2(H_n) \otimes \mathcal{D}_T^* \rightarrow F^2(H_n) \otimes \mathcal{D}_T,$$

with the formal Fourier representation

$$\Theta_T(R_1, \dots, R_n) := -I_{F^2(H_n)} \otimes T|_{\mathcal{D}_{T^*}} + (I_{F^2(H_n)} \otimes \Delta_T) \left(I_{F^2(H_n) \otimes \mathcal{K}} - \sum_{i=1}^n R_i \otimes T_i^* \right)^{-1} \\ [R_1 \otimes I_{\mathcal{K}}, \dots, R_n \otimes I_{\mathcal{K}}] (I_{F^2(H_n)} \otimes \Delta_{T^*}|_{\mathcal{D}_{T^*}}),$$

where R_1, \dots, R_n are the right creation operators on the full Fock space $F^2(H_n)$. Here, we need to clarify some notations since some of them are different from those considered in [5]. The defect operators associated with a row contraction $T := [T_1, \dots, T_n]$ are

$$\Delta_T := \left(I_{\mathcal{H}} - \sum_{i=1}^n T_i T_i^* \right)^{1/2} \in B(\mathcal{H}) \quad \text{and} \quad \Delta_{T^*} := (I - T^* T)^{1/2} \in B(\mathcal{H}^{(n)}),$$

while the defect spaces are $\mathcal{D}_T := \overline{\Delta_T \mathcal{H}}$ and $\mathcal{D}_{T^*} := \overline{\Delta_{T^*} \mathcal{H}^{(n)}}$. Using the F_n^∞ -functional calculus for row contractions [7], one can define

$$\Theta_T(X_1, \dots, X_n) := \text{SOT-}\lim_{r \rightarrow 1} \Theta_T(rX_1, \dots, rX_n)$$

for any c.n.c. row contraction $(X_1, \dots, X_n) \in [B(\mathcal{G})^n]_1^-$, where \mathcal{G} is a Hilbert space. Depending on T , the map Θ_T may be well-defined on a larger subset of $B(\mathcal{G})^n$. For example, if $\|T\| < 1$, then $X \mapsto \Theta_T(X)$ is a free holomorphic function on the open ball $[B(\mathcal{G})^n]_\gamma$, where $\gamma := \frac{1}{\|T\|}$. Therefore, the characteristic function $\tilde{\Theta}_T$ generates a bounded free holomorphic function Θ_T (also called characteristic function) on $[B(\mathcal{G})^n]_1$ with operator-valued coefficients in $B(\mathcal{D}_{T^*}, \mathcal{D}_T)$. Note also that

$$\Theta_T(X_1, \dots, X_n) = -I_{\mathcal{G}} \otimes (T|_{\mathcal{D}_{T^*}}) + (I_{\mathcal{G}} \otimes \Delta_T) \left(I_{\mathcal{G} \otimes \mathcal{K}} - \sum_{i=1}^n X_i \otimes T_i^* \right)^{-1} \\ [X_1 \otimes I_{\mathcal{K}}, \dots, X_n \otimes I_{\mathcal{K}}] (I_{\mathcal{G}} \otimes \Delta_{T^*}|_{\mathcal{D}_{T^*}})$$

for any $(X_1, \dots, X_n) \in [B(\mathcal{G})^n]_1$. The characteristic function $\tilde{\Theta}_T$ is the model boundary function of Θ_T with respect to R_1, \dots, R_n in the sense that

$$\tilde{\Theta}_T = \text{SOT-}\lim_{r \rightarrow 1} \Theta_T(rR_1, \dots, rR_n),$$

where $\Theta(rR_1, \dots, rR_n)$ is in $\mathcal{R}_n \otimes_{\min} B(\mathcal{K})$ for any $r \in [0, 1]$.

Let $T := [T_1, \dots, T_n]$ be a row contraction with $T_i \in B(\mathcal{H})$ and consider the subspace $\mathcal{H}_c \subseteq \mathcal{H}$ defined by

$$(1.1) \quad \mathcal{H}_c := \left\{ h \in \mathcal{H} : \sum_{|\alpha|=k} \|T_\alpha^* h\|^2 = \|h\|^2 \text{ for any } k = 1, 2, \dots \right\}.$$

We call T a *completely non-coisometric* (c.n.c.) row contraction if $\mathcal{H}_c = \{0\}$. We proved in [4] that \mathcal{H}_c is a joint invariant subspace under the operators T_1^*, \dots, T_n^* , and it is also the largest subspace in \mathcal{H} on which T^* acts isometrically. Consequently, we have the following triangulation with respect to the decomposition $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_{cnc}$:

$$T_i = \begin{pmatrix} A_i & 0 \\ * & B_i \end{pmatrix}, \quad i = 1, \dots, n,$$

where $*$ stands for an unspecified entry, $[A_1, \dots, A_n]$ is a coisometry, i.e., $A_1 A_1^* + \dots + A_n A_n^* = I_{\mathcal{H}_c}$, and $[B_1, \dots, B_n]$ is a c.n.c. row contraction. We say that a row contraction T is *pure* if

$$\lim_{k \rightarrow \infty} \sum_{\gamma \in \mathbb{F}_n^+, |\gamma|=k} \|T_\gamma^* h\|^2 = 0, \quad h \in \mathcal{H}.$$

An n -tuple $N := (N_1, \dots, N_n) \in B(\mathcal{H})^n$ is called *nilpotent* if there is $m \in \mathbb{N}$ such that $N_\alpha = 0$ for all $\alpha \in \mathbb{F}_n^+$ with $|\alpha| = m$. The order of a nilpotent n -tuple N is the smallest $m \in \mathbb{N}$ with the above-mentioned property. Throughout this paper, we make the convention that the degree of a constant polynomial (including the zero polynomial) is zero.

Theorem 1.1. *Let $T := [T_1, \dots, T_n] \in [B(\mathcal{H})^n]_1^-$ be a row contraction such that the characteristic function Θ_T is a noncommutative polynomial of degree $m \in \mathbb{N}$. Then there exist subspaces \mathcal{H}_v , \mathcal{H}_{nil} , and \mathcal{H}_c of \mathcal{H} such that $\mathcal{H} = \mathcal{H}_v \oplus \mathcal{H}_{nil} \oplus \mathcal{H}_c$ and each T_i admits a representation*

$$T_i = \begin{bmatrix} V_i & * & * \\ 0 & N_i & * \\ 0 & 0 & W_i \end{bmatrix}, \quad i = 1, \dots, n,$$

where $[V_1, \dots, V_n] \in [B(\mathcal{H}_v)^n]_1^-$ is a pure isometry, $[N_1, \dots, N_n] \in [B(\mathcal{H}_{nil})^n]_1^-$ is a nilpotent row contraction of order $\leq m$, and $[W_1, \dots, W_n] \in [B(\mathcal{H}_c)^n]_1^-$ is a coisometry. Moreover, if $m = 0$, then $\mathcal{H}_{nil} = \{0\}$ and T_i admits the representation

$$T_i = \begin{bmatrix} V_i & * \\ 0 & W_i \end{bmatrix}, \quad i = 1, \dots, n,$$

with respect to the decomposition $\mathcal{H} = \mathcal{H}_v \oplus \mathcal{H}_c$.

Proof. The characteristic function $\Theta_T : [B(\mathcal{G})^n]_1 \rightarrow B(\mathcal{G}) \bar{\otimes}_{\min} B(\mathcal{D}_{T^*}, \mathcal{D}_T)$ is a bounded free holomorphic function given by

$$\Theta_T(X_1, \dots, X_n) = -I_{\mathcal{G}} \otimes (T|_{\mathcal{D}_{T^*}}) + \sum_{i=1}^n \sum_{k=0}^{\infty} \sum_{|\alpha|=k} X_{\alpha} X_i \otimes \Delta_T(T_{\tilde{\alpha}})^* P_i \Delta_{T^*}|_{\mathcal{D}_{T^*}}$$

for $X = (X_1, \dots, X_n) \in [B(\mathcal{G})^n]_1$, where the convergence is in the operator norm and P_i denotes the orthogonal projection of $\mathcal{H}^{(n)}$ onto the i -component of $\mathcal{H}^{(n)}$. Assume that Θ_T is a noncommutative polynomial of degree $m \in \mathbb{N} = \{0, 1, \dots\}$. Then we have $\Delta_T(T_{\beta})^* P_i \Delta_{T^*} = 0$ for all $\beta \in \mathbb{F}_n^+$ with $|\beta| \geq m$ and $i = 1, \dots, n$. Hence, we deduce that

$$(1.2) \quad \Delta_{T^*}^2 J_i T_{\beta} \Delta_T = 0, \quad |\beta| \geq m, \quad i = 1, \dots, n,$$

where $J_i : \mathcal{H} \rightarrow \mathcal{H}^{(n)}$ is the injection $J_i h := \oplus_{j=1}^n \delta_{ji} h$. Define the subspace

$$\mathcal{H}_v := \overline{\text{span}}\{T_{\beta} h : h \in \mathcal{D}_T, |\beta| \geq m\}$$

and note that it is invariant under each operator T_1, \dots, T_n . In what follows, we show that the n -tuple $[T_1|_{\mathcal{H}_v}, \dots, T_n|_{\mathcal{H}_v}] \in [B(\mathcal{H}_v)^n]_1^-$ is an isometry. Note that if $h \in \mathcal{H}$, $|\beta| \geq m$, and $i = 1, \dots, n$, then relation (1.2) implies

$$\Delta_{T^*}^2 J_i T_{\beta} \Delta_T h = \begin{bmatrix} I - T_1^* T_1 & -T_1^* T_2 & \cdots & -T_1^* T_n \\ -T_2^* T_1 & I - T_2^* T_2 & \cdots & -T_2^* T_n \\ \vdots & \vdots & \ddots & \vdots \\ -T_i^* T_1 & -T_i^* T_2 & I - T_i^* T_i & -T_i^* T_n \\ \vdots & \vdots & \vdots & \vdots \\ -T_n^* T_1 & -T_n^* T_2 & \cdots & I - T_n^* T_n \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ T_{\beta} \Delta_T h \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} -T_1^* T_i T_{\beta} \Delta_T h \\ -T_2^* T_i T_{\beta} \Delta_T h \\ \vdots \\ (I - T_i^* T_i) T_{\beta} \Delta_T h \\ \vdots \\ -T_n^* T_i T_{\beta} \Delta_T h \end{bmatrix} = 0.$$

Consequently, we have

$$T_j^* T_i T_{\beta} \Delta_T h = 0, \quad i, j \in \{1, \dots, n\}, i \neq j \text{ and } |\beta| \geq m,$$

and

$$(I - T_i^* T_i) T_{\beta} \Delta_T h = 0, \quad i \in \{1, \dots, n\}.$$

Hence, we deduce that $T_i(\mathcal{H}_v) \perp T_j(\mathcal{H}_v)$ if $i \neq j$ and $\|T_i x\| = \|x\|$ for any $x \in \mathcal{H}_v$. Therefore, the n -tuple $[T_1|_{\mathcal{H}_v}, \dots, T_n|_{\mathcal{H}_v}] \in [B(\mathcal{H}_v)^n]_1^-$ is an isometry. Set $V_i := T_i|_{\mathcal{H}_v} : \mathcal{H}_v \rightarrow \mathcal{H}_v$ for $i = 1, \dots, n$. According to the Wold decomposition for isometries with orthogonal ranges (see [4]), there is a unique orthogonal decomposition $\mathcal{H}_v = \mathcal{H}_s \oplus \mathcal{H}_u$ such that \mathcal{H}_u and \mathcal{H}_s are reducing subspaces under V_1, \dots, V_n , the n -tuple $[V_1|_{\mathcal{H}_s}, \dots, V_n|_{\mathcal{H}_s}]$ is a pure row isometry and $[V_1|_{\mathcal{H}_u}, \dots, V_n|_{\mathcal{H}_u}]$ is a Cuntz isometry, i.e., $\sum_{i=1}^n (V_i|_{\mathcal{H}_u})(V_i|_{\mathcal{H}_u})^* = I_{\mathcal{H}_u}$. Moreover, we have

$$\mathcal{H}_u = \{h \in \mathcal{H}_v : \sum_{|\alpha|=k} \|V_{\alpha} h\|^2 = \|h\|^2 \text{ for all } k \in \mathbb{N}\}.$$

Note that, since $T = [T_1, \dots, T_n]$ is a row contraction, if $h \in \mathcal{H}_u$, then

$$\|h\|^2 = \sum_{|\alpha|=k} \|V_\alpha^* h\|^2 = \sum_{|\alpha|=k} \|P_{\mathcal{H}_v} T_\alpha^* h\|^2 \leq \sum_{|\alpha|=k} \|T_\alpha^* h\|^2 \leq \|h\|^2$$

for any $k \in \mathbb{N}$. Consequently, $\sum_{|\alpha|=k} \|T_\alpha^* h\|^2 = \|h\|^2$ for $k \in \mathbb{N}$, which proves that $h \in \mathcal{H}_c$. Therefore, we have $\mathcal{H}_u \subseteq \mathcal{H}_c$, where \mathcal{H}_c is given by relation (1.1).

Define the subspaces

$$\mathcal{M} := \overline{\text{span}}\{T_\alpha h : h \in \mathcal{D}_T, \alpha \in \mathbb{F}_n^+\}$$

and $\mathcal{H}_{nil} := \mathcal{M} \ominus \mathcal{H}_v$, and let $[N_1, \dots, N_n] \in B(\mathcal{H}_{nil})^n$ be the n -tuple of operators given by $N_i := P_{\mathcal{H}_{nil}} T_i|_{\mathcal{H}_{nil}}$ for $i = 1, \dots, n$. Since

$$\sum_{i=1}^n N_i N_i^* \leq P_{\mathcal{H}_{nil}} \left(\sum_{i=1}^n T_i T_i^* \right) |_{\mathcal{H}_{nil}} \leq I_{\mathcal{H}_{nil}},$$

we deduce that $[N_1, \dots, N_n] \in [B(\mathcal{H}_{nil})^n]_1^-$. Note that if $m = 0$, then $\mathcal{H}_{nil} = \{0\}$. On the other hand, since \mathcal{M} and \mathcal{H}_v are invariant subspaces under each operator T_1, \dots, T_n , the subspace \mathcal{H}_{nil} is semi-invariant under the same operators and, consequently, $N_\alpha = P_{\mathcal{H}_{nil}} T_\alpha|_{\mathcal{H}_{nil}}$ for all $\alpha \in \mathbb{F}_n^+$. Note that, due to the fact that $T_\beta \mathcal{M} \subseteq \mathcal{H}_v$ for all $\beta \in \mathbb{F}_n^+$ with $|\beta| \geq m$, we have $N_\beta = 0$ for $|\beta| \geq m$. Therefore, N is a nilpotent row contraction of order $\leq m$, and

$$T_i|_{\mathcal{H}_v \oplus \mathcal{H}_{nil}} = \begin{bmatrix} V_i & * \\ 0 & N_i \end{bmatrix}, \quad i = 1, \dots, n,$$

where $V_i := T_i|_{\mathcal{H}_v} : \mathcal{H}_v \rightarrow \mathcal{H}_v$.

Now, let $\mathcal{H}_3 := \mathcal{H} \ominus \mathcal{M}$ and define $W_i := P_{\mathcal{H}_3} T_i|_{\mathcal{H}_3}$ for $i = 1, \dots, n$. Note that a vector $h \in \mathcal{H}$ is in \mathcal{H}_3 if and only if $h \perp T_\alpha \Delta_T x$ for all $x \in \mathcal{H}$ and $\alpha \in \mathbb{F}_n^+$, which is equivalent to $\Delta_T T_\alpha^* h = 0$ for all $\alpha \in \mathbb{F}_n^+$. Consequently, $h \in \mathcal{H}_3$ if and only if

$$(I - T_1 T_1^* - \dots - T_n T_n^*) T_\alpha^* h = 0, \quad \alpha \in \mathbb{F}_n^+.$$

Therefore, if $h \in \mathcal{H}_3$, then one can prove by induction over $k \in \mathbb{N}$ that

$$\|h\|^2 = \sum_{i=1}^n \langle T_i T_i^* h, h \rangle = \sum_{|\alpha|=2} \langle T_\alpha T_\alpha^* h, h \rangle = \dots = \sum_{|\alpha|=k} \langle T_\alpha T_\alpha^* h, h \rangle$$

for all $k \in \mathbb{N}$. This shows that

$$\mathcal{H}_3 \subseteq \mathcal{H}_c := \left\{ h \in \mathcal{H} : \sum_{|\alpha|=k} \|T_\alpha^* h\|^2 = \|h\|^2 \text{ for any } k = 1, 2, \dots \right\}.$$

We prove now the reverse inclusion. Since $T_i^* \mathcal{H}_c \subseteq \mathcal{H}_c$ for $i = 1, \dots, n$, for any $h \in \mathcal{H}_c$ and $\beta \in \mathbb{F}_n^+$, we deduce that

$$\sum_{|\alpha|=k} \|T_\alpha^* T_\beta^* h\|^2 = \|T_\beta^* h\|^2, \quad k = 1, 2, \dots$$

In particular, we have $\langle T_\beta (I - \sum_{i=1}^n T_i T_i^*) T_\beta^* h, h \rangle = 0$, whence $\Delta_T T_\beta^* h = 0$ for all $\beta \in \mathbb{F}_n^+$. Therefore, $h \in \mathcal{H}_3$, which completes the proof of the fact that $\mathcal{H}_3 = \mathcal{H}_c$, the largest co-invariant subspace under

T_1, \dots, T_n such that $\begin{bmatrix} T_1^*|_{\mathcal{H}_c} \\ \vdots \\ T_n^*|_{\mathcal{H}_c} \end{bmatrix}$ is an isometry. This implies that $\sum_{i=1}^n W_i W_i^* = I_{\mathcal{H}_c}$. We have also seen that

$\mathcal{H}_u \subseteq \mathcal{H}_c = \mathcal{H}_3 := \mathcal{H} \ominus \mathcal{M}$ and $\mathcal{H}_u \subseteq \mathcal{H}_v \subseteq \mathcal{M}$. Consequently, $\mathcal{H}_u = \{0\}$ and T_i has the representation

$$T_i = \begin{bmatrix} V_i & * & * \\ 0 & N_i & * \\ 0 & 0 & W_i \end{bmatrix}, \quad i = 1, \dots, n,$$

where V , N , and W are n -tuples of operators with the required properties. If $m = 0$, then T_i admits the representation

$$T_i = \begin{bmatrix} V_i & * \\ 0 & W_i \end{bmatrix}, \quad i = 1, \dots, n,$$

with respect to the decomposition $\mathcal{H} = \mathcal{H}_v \oplus \mathcal{H}_c$. The proof is complete. \square

Theorem 1.2. *Let \mathcal{H}_0 , \mathcal{H}_1 , and \mathcal{H}_2 be Hilbert spaces and let V , N , and W be n -tuples of operators with the following properties:*

- (i) $V := [V_1, \dots, V_n] \in [B(\mathcal{H}_0)^n]_1^-$ is an isometry;
- (ii) $N := [N_1, \dots, N_n] \in [B(\mathcal{H}_1)^n]_1^-$ is a nilpotent row contraction of order $m \in \mathbb{N}$ with $\mathcal{H}_1 = \{0\}$ if $m = 0$;
- (iii) $W := [W_1, \dots, W_n] \in [B(\mathcal{H}_2)^n]_1^-$ is a coisometry.

Then the following statements hold.

- (a) If $m \geq 1$, then the characteristic function of any row contraction $[T_1, \dots, T_n] \in [B(\mathcal{H})^n]_1^-$ of the form

$$T_i = \begin{bmatrix} V_i & * & * \\ 0 & N_i & * \\ 0 & 0 & W_i \end{bmatrix}, \quad i = 1, \dots, n,$$

with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$, is a polynomial of degree $\leq m$.

- (b) If $m = 0$, then the characteristic function of any row contraction $[T_1, \dots, T_n] \in [B(\mathcal{H})^n]_1^-$ of the form

$$T_i = \begin{bmatrix} V_i & * \\ 0 & W_i \end{bmatrix}, \quad i = 1, \dots, n,$$

with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_2$, is a polynomial of degree zero.

Proof. First, we consider the case when $m \geq 1$. Since $V_i^* V_j = \delta_{ij} I$ for $i, j \in \{1, \dots, n\}$, we have

$$T_i^* T_j = \begin{bmatrix} \delta_{ij} I & * & * \\ * & * & * \\ * & * & * \end{bmatrix}, \quad i = 1, \dots, n,$$

where $*$ stands for an unspecified entry. Consequently, we deduce that $\Delta_{T^*}^2 = [\delta_{ij} I_{\mathcal{H}} - T_i^* T_j]_{n \times n} = [\mathbf{K}_{ij}]_{n \times n}$, where each operator entry $\mathbf{K}_{ij} \in B(\mathcal{H})$ has the form $\mathbf{K}_{ij} = [K_{ij}^{(pq)}]_{3 \times 3} = \begin{bmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$. Let Δ_{T^*} have the matrix representation $\Delta_{T^*} = [\mathbf{D}_{ij}]_{n \times n}$, where each entry \mathbf{D}_{ij} has the form $[D_{ij}^{(pq)}]_{3 \times 3}$, $p, q \in \{1, 2, 3\}$, with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$. Since Δ_{T^*} is a positive operator, we must have $\mathbf{D}_{ii} \geq 0$ and $\mathbf{D}_{ji} = \mathbf{D}_{ij}^*$ for all $i, j \in \{1, \dots, n\}$. This implies $D_{ii}^{(pp)} \geq 0$ for all $i \in \{1, \dots, n\}$ and $p \in \{1, 2, 3\}$, and $D_{ji}^{(qp)} = (D_{ij}^{(pq)})^*$ for all $i, j \in \{1, \dots, n\}$ and $p, q \in \{1, 2, 3\}$. Since $K_{ii}^{(11)} = 0$ and

$$K_{ii}^{(11)} = \sum_{q=1}^3 \sum_{j=1}^n D_{ij}^{(1q)} (D_{ij}^{(1q)})^* \quad \text{for } i \in \{1, \dots, n\},$$

we deduce that $D_{ij}^{(1q)} = 0$ for all $i, j \in \{1, \dots, n\}$ and $q \in \{1, 2, 3\}$. Therefore Δ_{T^*} has the operator matrix representation

$$\Delta_{T^*} = \begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix} & \cdots & \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \\ \vdots & \ddots & \vdots \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix} & \cdots & \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \end{bmatrix}.$$

Now, note that

$$\Delta_T^2 = I - \sum_{i=1}^n T_i T_i^* = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & 0 \end{bmatrix}.$$

Setting $\Delta_T = [\Lambda_{pq}]_{3 \times 3}$ and taking into account that $\Delta_T \geq 0$, we deduce that $\Lambda_{pp} \geq 0$ and $\Lambda_{qp} = \Lambda_{pq}^*$ for $p, q \in \{1, 2, 3\}$. Since

$$\Delta_T^2 = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & \Lambda_{13}^* \Lambda_{13} + \Lambda_{23}^* \Lambda_{23} + \Lambda_{33}^2 \end{bmatrix},$$

we must have $\Lambda_{13}^* \Lambda_{13} + \Lambda_{23}^* \Lambda_{23} + \Lambda_{33}^2 = 0$, which implies $\Lambda_{13} = \Lambda_{23} = \Lambda_{33} = 0$. Therefore, Δ_T has the form

$$\Delta_T = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$. Since

$$T_\beta = \begin{bmatrix} V_\beta & * & * \\ 0 & 0 & * \\ 0 & 0 & W_\beta \end{bmatrix}$$

for all $\beta \in \mathbb{F}_n^+$ with $|\beta| \geq m \geq 1$, we deduce that

$$\begin{aligned} \Delta_T T_\beta^* P_i \Delta_T^* (\oplus_{i=1}^n h_i) &= \sum_{i=1}^n \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_\beta^* & 0 & 0 \\ * & 0 & 0 \\ * & * & W_\beta^* \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix} h_i \\ &= \sum_{i=1}^n \begin{bmatrix} * & 0 & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix} h_i = 0 \end{aligned}$$

for any $\oplus_{i=1}^n h_i \in \mathcal{H}^{(n)}$. Hence, $\Delta_T T_\beta^* P_i \Delta_T^* = 0$ for all $\beta \in \mathbb{F}_n^+$ with $|\beta| \geq m \geq 1$, which shows that the characteristic function Θ_T is a polynomial of degree $\leq m$.

Now, we consider the case when $m = 0$. Similar considerations as above reveal that Δ_{T^*} and Δ_T have the forms

$$\Delta_{T^*} = \begin{pmatrix} \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix} & \cdots & \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix} \\ \vdots & \vdots & \vdots \\ \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix} & \cdots & \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix} \end{pmatrix}$$

and $\Delta_T = \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_2$. Since $T_\beta = \begin{bmatrix} V_\beta & * \\ 0 & W_\beta \end{bmatrix}$ for all $\beta \in \mathbb{F}_n^+$, we have

$$\begin{aligned} \Delta_T T_\beta^* P_i \Delta_T^* (\oplus_{i=1}^n h_i) &= \sum_{i=1}^n \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_\beta^* & 0 \\ * & W_\beta^* \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix} h_i \\ &= \sum_{i=1}^n \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix} h_i = 0 \end{aligned}$$

for any $\oplus_{i=1}^n h_i \in \mathcal{H}^{(n)}$ and $\beta \in \mathbb{F}_n^+$. Hence, we deduce that the characteristic function Θ_T is a constant, i.e., $\Theta_T = \Theta_T(0)$. The proof is complete. \square

Combining Theorem 1.1 and Theorem 1.2, we obtain the following characterization for row contractions with polynomial characteristic functions.

Theorem 1.3. *Let $T := [T_1, \dots, T_n] \in [B(\mathcal{H})^n]_1^-$ be a row contraction. Then the characteristic function Θ_T is a noncommutative polynomial of degree $m \in \mathbb{N}$ if and only if there exist subspaces \mathcal{H}_0 , \mathcal{H}_1 , and \mathcal{H}_2 of \mathcal{H} such that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ and each T_i admits a representation*

$$T_i = \begin{bmatrix} V_i & * & * \\ 0 & N_i & * \\ 0 & 0 & W_i \end{bmatrix}, \quad i = 1, \dots, n,$$

where $V := [V_1, \dots, V_n] \in [B(\mathcal{H}_0)^n]_1^-$ is a pure isometry, $N := [N_1, \dots, N_n] \in [B(\mathcal{H}_1)^n]_1^-$ is a nilpotent row contraction of order m , and $W := [W_1, \dots, W_n] \in [B(\mathcal{H}_2)^n]_1^-$ is a coisometry. Moreover, the degree of Θ_T is the smallest possible order of N in the representation of T .

In general, a row contraction has many representations in upper triangular form. The next result shows that, in a certain sense, the representation provided by Theorem 1.1 is unique.

Proposition 1.4. *Let $T := [T_1, \dots, T_n] \in [B(\mathcal{H})^n]_1^-$ be a row contraction such that the characteristic function Θ_T is a noncommutative polynomial of degree $m \in \mathbb{N}$. Let $T := [T_1, \dots, T_n]$ have a representation*

$$T_i = \begin{bmatrix} V'_i & * & * \\ 0 & N'_i & * \\ 0 & 0 & W'_i \end{bmatrix}, \quad i = 1, \dots, n,$$

with respect to a decomposition $\mathcal{H} = \mathcal{H}'_0 \oplus \mathcal{H}'_1 \oplus \mathcal{H}'_2$, where $[V'_1, \dots, V'_n] \in [B(\mathcal{H}'_0)^n]_1^-$ is an isometry, $[N'_1, \dots, N'_n] \in [B(\mathcal{H}'_1)^n]_1^-$ is a nilpotent row contraction of order m , and $[W'_1, \dots, W'_n] \in [B(\mathcal{H}'_2)^n]_1^-$ is a coisometry.

Then the upper triangular representation of T given by Theorem 1.1 has the following properties: $\mathcal{H}_v \subseteq \mathcal{H}'_0$, $\mathcal{H}_c \supseteq \mathcal{H}'_2$. Moreover,

$$\begin{aligned} \mathcal{H}_v &= \overline{\text{span}}\{T_\beta h : h \in \mathcal{D}_T, |\beta| \geq m\}, \\ \mathcal{H}_c &= \{h \in \mathcal{H} : \sum_{|\alpha|=k} \|T_\alpha^* h\|^2 = \|h\|^2 \text{ for all } k \in \mathbb{N}\}, \quad \text{and} \\ \mathcal{H}_{\text{nil}} &= \mathcal{H} \ominus (\mathcal{H}_v \oplus \mathcal{H}_c). \end{aligned}$$

Proof. As in the proof of Theorem 1.2, we deduce that Δ_T has the form

$$\Delta_T = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with respect to the decomposition $\mathcal{H} = \mathcal{H}'_0 \oplus \mathcal{H}'_1 \oplus \mathcal{H}'_2$ and

$$T_\alpha \Delta_T = \begin{bmatrix} V'_\beta & * & * \\ 0 & 0 & * \\ 0 & 0 & W'_\beta \end{bmatrix} \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} * & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for any $\alpha \in \mathbb{F}_n^+$ with $|\alpha| \geq m$. Consequently, if $h = h_0 \oplus h_1 \oplus h_2$, where $h_j \in \mathcal{H}'_j$ for $j = 0, 1, 2$, then

$T_\alpha \Delta h = \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix} \in \mathcal{H}'_0$ for $|\alpha| \geq m$. Hence $\mathcal{H}_v \subset \mathcal{H}'_0$. Note that the inclusion $\mathcal{H}'_2 \subseteq \mathcal{H}_c$ is true due to the

fact that \mathcal{H}_c is the largest invariant subspace under T_1^*, \dots, T_n^* such that $\begin{bmatrix} T_1^*|_{\mathcal{H}_c} \\ \vdots \\ T_n^*|_{\mathcal{H}_c} \end{bmatrix}$ is an isometry. The

last part of the proposition follows from the proof of Theorem 1.1. \square

In what follows, we call the upper triangular representation of T given by Theorem 1.1 canonical.

We recall that a row contraction $T := [T_1, \dots, T_n] \in [B(\mathcal{H})^n]_1^-$ is called *completely non-unitary* (c.n.u.) if there is no nonzero subspace $\mathcal{M} \subseteq \mathcal{H}$ reducing under T_1, \dots, T_n such that $[T_1|_{\mathcal{M}}, \dots, T_n|_{\mathcal{M}}]$ is a unitary operator from $\mathcal{M}^{(m)}$ to \mathcal{M} .

Using Theorem 1.1, one can easily deduce the following

Corollary 1.5. *Let $T := [T_1, \dots, T_n] \in [B(\mathcal{H})^n]_1^-$ be a c.n.u. row contraction. Then the characteristic function Θ_T is a noncommutative polynomial of degree $m \in \mathbb{N}$ if and only if there exist subspaces \mathcal{H}_0 , \mathcal{H}_1 , and \mathcal{H}_2 of \mathcal{H} such that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ and each T_i admits a representation*

$$T_i = \begin{bmatrix} V_i & * & * \\ 0 & N_i & * \\ 0 & 0 & C_i \end{bmatrix}, \quad i = 1, \dots, n,$$

where $V := [V_1, \dots, V_n] \in [B(\mathcal{H}_0)^n]_1^-$ is a pure row isometry, $N := [N_1, \dots, N_n] \in [B(\mathcal{H}_1)^n]_1^-$ is a nilpotent row contraction of order m , and $C := [C_1, \dots, C_n] \in [B(\mathcal{H}_2)^n]_1^-$ is a c.n.u. coisometry.

We remark that there is a canonical upper triangular representation for c.n.u. row contractions, namely, the one provided by Theorem 1.1.

2. UNITARY INVARIANTS ON THE UNIT BALL OF $B(\mathcal{H})^n$

In general, a row contraction has many representations in upper triangular form. The next result gives another reason why we will focus on the canonical upper triangular representations of row contractions with polynomial characteristic functions.

Proposition 2.1. *Let $T := [T_1, \dots, T_n] \in [B(\mathcal{H})^n]_1^-$ and $T' := [T'_1, \dots, T'_n] \in [B(\mathcal{H}')^n]_1^-$ be row contractions with polynomial characteristic functions, and let*

$$T_i = \begin{bmatrix} V_i & * & * \\ 0 & N_i & * \\ 0 & 0 & W_i \end{bmatrix} \quad \text{and} \quad T'_i = \begin{bmatrix} V'_i & * & * \\ 0 & N'_i & * \\ 0 & 0 & W'_i \end{bmatrix}$$

be their canonical representations on $\mathcal{H} = \mathcal{H}_v \oplus \mathcal{H}_{nil} \oplus \mathcal{H}_c$ and $\mathcal{H}' = \mathcal{H}'_v \oplus \mathcal{H}'_{nil} \oplus \mathcal{H}'_c$, respectively. If $U : \mathcal{H} \rightarrow \mathcal{H}'$ is a unitary operator such that $UT_i = T'_i U$ for all $i = 1, \dots, n$, then

$$U(\mathcal{H}_v) = \mathcal{H}'_v, \quad U(\mathcal{H}_{nil}) = \mathcal{H}'_{nil}, \quad U(\mathcal{H}_c) = \mathcal{H}'_c,$$

and the diagonal entries of T and T' are unitarily equivalent, i.e.,

$$(U|_{\mathcal{H}_v})V_i = V'_i(U|_{\mathcal{H}_v}), \quad (U|_{\mathcal{H}_{nil}})N_i = N'_i(U|_{\mathcal{H}_{nil}}), \quad (U|_{\mathcal{H}_c})W_i = W'_i(U|_{\mathcal{H}_c})$$

for all $i = 1, \dots, n$. Moreover, if $T := [T_1, \dots, T_n]$ has a representation

$$T_i = \begin{bmatrix} A_i & * & * \\ 0 & B_i & * \\ 0 & 0 & C_i \end{bmatrix}, \quad i = 1, \dots, n,$$

with respect to a decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$, where $[A_1, \dots, A_n] \in [B(\mathcal{H}_0)^n]_1^-$ is a pure isometry, $[B_1, \dots, B_n] \in [B(\mathcal{H}_1)^n]_1^-$ is a nilpotent row contraction of order $m \in \mathbb{N}$, and $[C_1, \dots, C_n] \in [B(\mathcal{H}_2)^n]_1^-$ is a coisometry, then the diagonal entries of T are not, in general, unitarily equivalent with those corresponding to the canonical representation of T .

Proof. According to Section 1, we have

$$\mathcal{H}_v = \overline{\text{span}}\{T_\beta h : h \in \mathcal{D}_T, |\beta| \geq m\},$$

$$\mathcal{H}_c = \{h \in \mathcal{H} : \sum_{|\alpha|=k} \|T_\alpha^* h\|^2 = \|h\|^2 \text{ for all } k \in \mathbb{N}\},$$

$$\mathcal{H}_{nil} = \mathcal{H} \ominus (\mathcal{H}_v \oplus \mathcal{H}_c),$$

and similar formulas hold for \mathcal{H}'_v , \mathcal{H}'_c and \mathcal{H}'_{nil} , respectively. If $U : \mathcal{H} \rightarrow \mathcal{H}'$ is a unitary operator such that $UT_i = T'_i U$ for $i = 1, \dots, n$, then $U\Delta_T = \Delta_{T'}U$ and $U(\mathcal{H}_v) = \mathcal{H}'_v$, $U(\mathcal{H}_{nil}) = \mathcal{H}'_{nil}$, and $U(\mathcal{H}_c) = \mathcal{H}'_c$. Now, it is easy to see that the diagonal entries of T and T' are unitarily equivalent.

To prove the last part of the proposition, let \mathcal{N} be a separable Hilbert space and let $C_i \in B(\mathcal{N})$ be such that $C = [C_1, \dots, C_n]$ is a coisometry. Fix $m \geq 1$ and denote by \mathcal{P}_{m-1} the subspace of all polynomials

of degree $\leq m-1$ in the full Fock space $F^2(H_n)$, i.e. $\mathcal{P}_{m-1} := \text{span}\{e_\alpha : \alpha \in \mathbb{F}_n^+, |\alpha| \leq m-1\}$. Let $T := [T_1, \dots, T_n]$ be defined by

$$T_i = \begin{bmatrix} S_i & 0 & 0 \\ 0 & P_{\mathcal{P}_{m-1}} S_i|_{\mathcal{P}_{m-1}} & 0 \\ 0 & 0 & C_i \end{bmatrix}, \quad i = 1, \dots, n,$$

with respect to the decomposition $\mathcal{H} := F^2(H_n) \oplus \mathcal{P}_{m-1} \oplus \mathcal{N}$, where S_1, \dots, S_n are the left creation operators on $F^2(H_n)$. According to Theorem 1.1, the canonical decomposition of T_i is

$$T_i = \begin{bmatrix} V_i & * & * \\ 0 & N_i & * \\ 0 & 0 & C_i \end{bmatrix}, \quad i = 1, \dots, n,$$

with respect to the decomposition $\mathcal{H} = \mathcal{H}_v \oplus \mathcal{H}_{nil} \oplus \mathcal{H}_c$, where

$$\mathcal{H}_v := [F^2(H_n) \ominus \mathcal{P}_{m-1}] \oplus 0 \oplus 0, \quad \mathcal{H}_{nil} := \mathcal{P}_{m-1} \oplus \mathcal{P}_{m-1} \oplus 0, \quad \text{and} \quad \mathcal{H}_c := 0 \oplus 0 \oplus \mathcal{N},$$

the operators $V_i \in B(F^2(H_n) \ominus \mathcal{P}_{m-1})$, $N_i \in B(\mathcal{P}_{m-1} \oplus \mathcal{P}_{m-1})$, and $C_i \in B(\mathcal{L})$ are defined by

$$V_i := S_i|_{F^2(H_n) \ominus \mathcal{P}_{m-1}}, \quad N_i := \begin{bmatrix} P_{\mathcal{P}_{m-1}} S_i|_{\mathcal{P}_{m-1}} & 0 \\ 0 & P_{\mathcal{P}_{m-1}} S_i|_{\mathcal{P}_{m-1}} \end{bmatrix}, \quad \text{and} \quad W_i := C_i$$

for any $i = 1, \dots, n$. We remark that the pure isometries $[S_1, \dots, S_n]$ and $[V_1, \dots, V_n]$ are not unitarily equivalent, when $n \geq 2$, since they have the multiplicity 1 and n^m , respectively. Note also that the nilpotent row contractions $[P_{\mathcal{P}_{m-1}} S_1|_{\mathcal{P}_{m-1}}, \dots, P_{\mathcal{P}_{m-1}} S_n|_{\mathcal{P}_{m-1}}]$ and $[N_1, \dots, N_n]$ are not unitarily equivalent, in spite of having the same order m . The proof is complete. \square

We need to recall from [10] that the noncommutative Poisson kernel associated with a row contraction $T := [T_1, \dots, T_n] \in [B(\mathcal{H})^n]_1^-$ is the operator $K_T : \mathcal{H} \rightarrow F^2(H_n) \otimes \overline{\Delta_T} \mathcal{H}$ defined by

$$K_T h := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} e_\alpha \otimes \Delta_T T_\alpha^* h, \quad h \in \mathcal{H}.$$

The operator K_{rT} is an isometry if $0 < r < 1$, and

$$K_T^* K_T = I - \text{SOT-} \lim_{k \rightarrow \infty} \sum_{|\alpha|=k} T_\alpha T_\alpha^*.$$

The connection between the characteristic function and the Poisson kernel of a row contraction is given by the formula $I - \Theta_T \Theta_T^* = K_T K_T^*$ (see [13]).

Let $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$ and define the map

$$\Gamma : [B(\mathcal{H})^n]_1^- \rightarrow \mathbb{N}_\infty \times \mathbb{N}_\infty \times \mathbb{N}_\infty, \quad \Gamma(T) := (p, m, q),$$

by setting $m := \deg(\Theta_T)$, $q := \dim(\ker K_T)$, and

$$p := \begin{cases} \dim(\mathcal{D}_m \ominus \mathcal{D}_{m+1}) & \text{if } m \in \mathbb{N} \\ \dim \overline{\Delta_T} \mathcal{H} & \text{if } m = \infty, \end{cases}$$

where $\mathcal{D}_m := \overline{\text{span}}\{T_\beta \Delta_T h : h \in \mathcal{H}, |\beta| \geq m\}$, Θ_T is the characteristic function, K_T is the noncommutative Poisson kernel, and Δ_T is the defect operator associated with $T \in [B(\mathcal{H})^n]_1^-$. One can easily show that the map Γ is a unitary invariant for row contractions, i.e., if $T \in [B(\mathcal{H})^n]_1^-$ and $T' \in [B(\mathcal{H}')^n]_1^-$ are unitarily equivalent, then $\Gamma(T) = \Gamma(T')$.

The next result shows that the map Γ detects the pure row isometries in the closed unit ball of $B(\mathcal{H})^n$ and completely classify them up to a unitary equivalence.

Theorem 2.2. *Let $T := [T_1, \dots, T_n] \in [B(\mathcal{H})^n]_1^-$ be a row contraction. Then the following statements hold:*

- (i) *T is a pure isometry if and only if $\Gamma(T) \in \mathbb{N}_\infty \times \{0\} \times \{0\}$.*
- (ii) *If $T, T' \in [B(\mathcal{H})^n]_1^-$ and $\Gamma(T) = \Gamma(T') = (p, 0, 0)$ for some $p \in \mathbb{N}_\infty$, then T is unitarily equivalent to T' and $p = \text{rank } \Delta_T = \text{rank } \Delta_{T'}$.*

Proof. First, we assume that T is a pure isometry. According to the Wold decomposition for isometries with orthogonal subspaces [4], T is unitarily equivalent to $(S_1 \otimes I_{\mathcal{K}}, \dots, S_n \otimes I_{\mathcal{K}})$ for some Hilbert space \mathcal{K} . Therefore, without loss of generality we can assume that $T = [S_1 \otimes I_{\mathcal{K}}, \dots, S_n \otimes I_{\mathcal{K}}]$. In this case, we have $\Delta_T = P_{\mathbb{C}} \otimes I_{\mathcal{K}}$ and $\Delta_{T^*} = 0$. Consequently, we deduce that $\mathcal{D}_T = 1 \otimes \mathcal{K}$, $\mathcal{D}_{T^*} = \{0\}$, and $\Theta_T = 0 \in B(\{0\}, \mathcal{K})$. Hence, $\deg(\Theta_T) = 0$. Note that

$$\overline{\text{span}}\{T_{\beta}\Delta_T h : h \in \mathcal{H}, |\beta| \geq m\} \ominus \overline{\text{span}}\{T_{\beta}\Delta_T h : h \in \mathcal{H}, |\beta| \geq m+1\} = 1 \otimes \mathcal{K}$$

and $p = \dim \mathcal{K} = \text{rank } \Delta_T$. On the other hand, since

$$\ker K_T = \{h \in \mathcal{H} : \sum_{|\alpha|=k} \|T_{\alpha}^* h\|^2 = \|h\|^2 \text{ for all } k \in \mathbb{N}\} = \mathcal{K}_c$$

and $T = [S_1 \otimes I_{\mathcal{K}}, \dots, S_n \otimes I_{\mathcal{K}}]$ is a completely non-coisometric row contraction, we have $\ker K_T = \mathcal{H}_c = \{0\}$ and, therefore, $\dim \ker K_T = 0$. Summing up, we deduce that $\Gamma(T) \in \mathbb{N}_{\infty} \times \{0\} \times \{0\}$.

Conversely, assume that T is a row contraction with $\Gamma(T) \in \mathbb{N}_{\infty} \times \{0\} \times \{0\}$. Then we have $\ker K_T = \mathcal{H}_c = \{0\}$. According to Theorem 1.1, T_i admits the representation

$$T_i = \begin{bmatrix} V_i & * \\ 0 & N_i \end{bmatrix}, \quad i = 1, \dots, n,$$

with respect to the decomposition $\mathcal{H} = \mathcal{H}_v \oplus \mathcal{H}_{nil}$. On the other hand, since $\deg(\Theta_T) = 0$, we must have $\mathcal{H}_{nil} = \{0\}$ and $T_i = V_i$ for $i = 1, \dots, n$. Therefore, $T = [V_1, \dots, V_n]$ is a pure isometry on \mathcal{H} and, using the Wold decomposition for isometries with orthogonal subspaces, we deduce that

$$p = \dim [\overline{\text{span}}\{T_{\beta}h : h \in \mathcal{D}_T, \beta \in \mathbb{F}_n^+\} \ominus \overline{\text{span}}\{T_{\beta}h : h \in \mathcal{D}_T, |\beta| \geq 1\}]$$

is the dimension of the wandering subspace for $T = [V_1, \dots, V_n]$. Hence, $T = [V_1, \dots, V_n]$ is unitarily equivalent to $[S_1 \otimes I_{\mathcal{K}}, \dots, S_n \otimes I_{\mathcal{K}}]$ for some Hilbert space \mathcal{K} with $\dim \mathcal{K} = p$, where S_1, \dots, S_n are the left creation operators on the full Fock space $F^2(H_n)$. Therefore, part (i) holds.

To prove part (ii), assume that $T, T' \in [B(\mathcal{H})^n]_1^-$ and $\Gamma(T) = \Gamma(T') = (p, 0, 0)$ for some $p \in \mathbb{N}_{\infty}$. Due to the first part of the proof, we deduce that T and T' are pure row contractions with the property that the dimensions of their wandering subspaces are equal to $p = \text{rank } \Delta_T = \text{rank } \Delta_{T'}$. Consequently, using the Wold decomposition, we conclude that the pure row isometries T and T' are unitarily equivalent. The proof is complete. \square

We remark that, due to Theorem 2.2 and the model theory for row contraction [5], if $q = 0$ and $m = 0$ or $q = 0$ and $m = \infty$, then p represents the multiplicity of the n -tuple (S_1, \dots, S_n) of left creation operators in the operator model of $T = (T_1, \dots, T_n)$.

Corollary 2.3. *Let $T := (T_1, \dots, T_n) \in [B(\mathcal{H})^n]_1^-$ be a row contraction and S_1, \dots, S_n be the left creation operators on the full Fock space $F^2(H_n)$. Then T is unitarily equivalent to $(S_1 \otimes I_{\mathcal{K}}, \dots, S_n \otimes I_{\mathcal{K}})$ for some Hilbert space \mathcal{K} if and only if*

$$\Gamma(T) = (\dim \mathcal{K}, 0, 0).$$

In this case, $\text{rank } \Delta_T = \dim \mathcal{K}$.

Let $\Phi : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes} B(\mathcal{K}_1, \mathcal{K}_2)$ and $\Phi' : [B(\mathcal{H})^n]_1 \rightarrow B(\mathcal{H}) \bar{\otimes} B(\mathcal{K}'_1, \mathcal{K}'_2)$ be two free holomorphic functions. We say that Φ and Φ' coincide if there are two unitary operators $\tau_j \in B(\mathcal{K}_j, \mathcal{K}'_j)$, $j = 1, 2$, such that

$$\Phi'(X)(I_{\mathcal{H}} \otimes \tau_1) = (I_{\mathcal{H}} \otimes \tau_2)\Phi(X), \quad X \in [B(\mathcal{H})^n]_1.$$

Now, we can prove the following classification result.

Theorem 2.4. *Let $T := [T_1, \dots, T_n] \in [B(\mathcal{H})^n]_1^-$ be a row contraction. Then the following statements hold:*

- (i) *T is a pure row contraction with polynomial characteristic function if and only if*

$$\Gamma(T) \in \mathbb{N}_{\infty} \times \mathbb{N} \times \{0\}.$$

- In this case, T_i has the canonical form $T_i = \begin{bmatrix} V_i & * \\ 0 & N_i \end{bmatrix}$, where $V := [V_1, \dots, V_n] \in [B(\mathcal{H}_v)^n]_1^-$ is a pure isometry and $N := [N_1, \dots, N_n] \in [B(\mathcal{H}_{nil})^n]_1^-$ is a nilpotent row contraction.
- (ii) the map $T \mapsto (\Gamma(T), \Theta_T)$ detects the pure row contractions with polynomial characteristic functions and completely classify them.

Proof. Assume that T is a pure row contraction with polynomial characteristic function. Let

$$T_i = \begin{bmatrix} V_i & * & * \\ 0 & N_i & * \\ 0 & 0 & W_i \end{bmatrix}$$

be the canonical upper triangular representation on $\mathcal{H} = \mathcal{H}_v \oplus \mathcal{H}_{nil} \oplus \mathcal{H}_c$, provided by Theorem 1.1. If $h \in \mathcal{H}_c$, then $T_\alpha^*(0 \oplus 0 \oplus h) = 0 \oplus 0 \oplus W_\alpha^*h$. Consequently, we have

$$\|h\|^2 = \sum_{|\alpha|=k} \|W_\alpha^*h\|^2 = \sum_{|\alpha|=k} \|T_\alpha^*(0 \oplus 0 \oplus h)\|^2, \quad k \in \mathbb{N}.$$

Since T is a pure row contraction, we deduce that $h = 0$, which shows that $\mathcal{H}_c = \{0\}$. Therefore, $\Gamma(T) \in \mathbb{N}_\infty \times \mathbb{N} \times \{0\}$ and T_i has the form $T_i = \begin{bmatrix} V_i & * \\ 0 & N_i \end{bmatrix}$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_v \oplus \mathcal{H}_{nil}$, where $V := [V_1, \dots, V_n] \in [B(\mathcal{H}_v)^n]_1^-$ is a pure isometry and $N := [N_1, \dots, N_n] \in [B(\mathcal{H}_{nil})^n]_1^-$ is a nilpotent row contraction.

Conversely, assume that T is a row contraction with $\Gamma(T) \in \mathbb{N}_\infty \times \mathbb{N} \times \{0\}$. Hence, $\dim(\ker K_T) = 0$ and $\mathcal{H}_c = \{0\}$. According to Theorem 1.1, T_i has the form $T_i = \begin{bmatrix} V_i & * \\ 0 & N_i \end{bmatrix}$. Assuming that N is a nilpotent n -tuple of order m , we deduce that there exist operators $X_{(\alpha)} \in B(\mathcal{H}_{nil}, \mathcal{H}_v)$ such that

$$(2.1) \quad T_\alpha = \begin{bmatrix} V_\alpha & X_{(\alpha)} \\ 0 & 0 \end{bmatrix}$$

for all $\alpha \in \mathbb{F}_n^+$ with $|\alpha| = m$. Since $[T_1, \dots, T_n]$ is a row contraction, so is the row operator $[T_\alpha : |\alpha| = k]$ for any $k \geq 1$. In particular, we have

$$\sum_{|\alpha|=m} \|T_\alpha^*(x \oplus 0)\|^2 = \sum_{|\alpha|=m} \|V_\alpha^*x\|^2 + \sum_{|\alpha|=m} \|X_{(\alpha)}^*x\|^2 \leq \|x\|^2$$

for any $x \in \mathcal{H}_v$. Consequently, the row operator $[X_{(\alpha)} : |\alpha| = m]$ is a contraction. Let $\alpha_1, \dots, \alpha_k \in \mathbb{F}_n^+$ be such that $|\alpha_1| = \dots = |\alpha_k| = m$, and note that, due to relation (2.1),

$$T_{\alpha_1} \dots T_{\alpha_k} = \begin{bmatrix} V_{\alpha_1} \dots V_{\alpha_k} & V_{\alpha_1} \dots V_{\alpha_{k-1}} X_{(\alpha_k)} \\ 0 & 0 \end{bmatrix}.$$

Since $[X_{(\alpha)} : |\alpha| = m]$ is a contraction, we have

$$\begin{aligned} \sum_{\substack{\alpha_1, \dots, \alpha_k \in \mathbb{F}_n^+ \\ |\alpha_1| = \dots = |\alpha_k| = m}} \|T_{\alpha_1 \dots \alpha_k}^*(x \oplus y)\|^2 &= \sum_{\substack{\alpha_1, \dots, \alpha_k \in \mathbb{F}_n^+ \\ |\alpha_1| = \dots = |\alpha_k| = m}} \|V_{\alpha_1 \dots \alpha_k}^*x\|^2 + \sum_{\substack{\alpha_1, \dots, \alpha_k \in \mathbb{F}_n^+ \\ |\alpha_1| = \dots = |\alpha_k| = m}} \|X_{\alpha_k}^* V_{\alpha_1 \dots \alpha_{k-1}}^*x\|^2 \\ &\leq \sum_{\gamma \in \mathbb{F}_n^+, |\gamma|=mk} \|V_\gamma^*x\|^2 + \sum_{\gamma \in \mathbb{F}_n^+, |\gamma|=m(k-1)} \|V_\gamma^*x\|^2 \end{aligned}$$

for any $x \oplus y \in \mathcal{H}_v \oplus \mathcal{H}_{nil}$. Taking into account that $[V_1, \dots, V_n]$ is a pure isometry, we have

$$\lim_{k \rightarrow \infty} \sum_{\gamma \in \mathbb{F}_n^+, |\gamma|=k} \|V_\gamma^*x\|^2 = 0, \quad x \in \mathcal{H}_v.$$

Hence, and using the inequalities above, we conclude that

$$(2.2) \quad \lim_{k \rightarrow \infty} \sum_{\gamma \in \mathbb{F}_n^+, |\gamma|=mk} \|T_\gamma^*(x \oplus y)\|^2 = 0, \quad x \oplus y \in \mathcal{H}_v \oplus \mathcal{H}_{nil}.$$

If $q \geq \mathbb{N}$, then $q = mk_q + p_q$ for unique $k_q \in \mathbb{N}$ and $p_q \in \{0, 1, \dots, m-1\}$. Using the fact that $[T_\gamma : |\gamma| = p_q]$ is a row contraction, we have

$$\begin{aligned} \sum_{|\alpha|=q} \|T_\gamma^*(x \oplus y)\|^2 &= \sum_{|\gamma|=p_q, |\sigma|=mk_q} \|T_\gamma^* T_\sigma^*(x \oplus y)\|^2 \\ &\leq \sum_{|\sigma|=mk_q} \|T_\sigma^*(x \oplus y)\|^2. \end{aligned}$$

Hence, and using (2.2), we deduce that $\lim_{p \rightarrow \infty} \sum_{|\alpha|=q} \|T_\gamma^*(x \oplus y)\|^2 = 0$, which proves that $[T_1, \dots, T_n]$ is a pure row contraction. The proof of part (i) complete.

To prove part (ii), let $T, T' \in [B(\mathcal{H})^n]_1^-$ be row contractions. Using the result from part (i) and Theorem 5.4 from [5], we deduce that T and T' are unitarily equivalent pure row contractions with polynomial characteristic functions if and only if $\Gamma(T)$ and $\Gamma(T')$ are in $\mathbb{N}_\infty \times \mathbb{N} \times \{0\}$, and the characteristic functions Θ_T and $\Theta_{T'}$ coincide. This completes the proof. \square

Using Theorem 1.3, we can easily deduce the following

Proposition 2.5. *Let $T := [T_1, \dots, T_n] \in [B(\mathcal{H})^n]_1^-$ be a row contraction with polynomial characteristic function. Then the following statements hold.*

- (i) T_i has the form $T_i = \begin{bmatrix} N_i & * \\ 0 & W_i \end{bmatrix}$ if and only if $\Gamma(T) \in \{0\} \times \mathbb{N} \times \mathbb{N}_\infty$.
- (ii) T_i has the form $[N_i]$ if and only if $\Gamma(T) \in \{0\} \times \mathbb{N} \times \{0\}$.
- (iii) T_i has the form $[W_i]$ if and only if $\Gamma(T) \in \{0\} \times \{0\} \times \mathbb{N}_\infty$.

Corollary 2.6. *Let $T := [T_1, \dots, T_n] \in [B(\mathcal{H})^n]_1^-$ be a row contraction. Then T is c.n.c. if and only if $\Gamma(T) \in \mathbb{N}_\infty \times \mathbb{N}_\infty \times \{0\}$. In this case, the characteristic function Θ_T is a noncommutative polynomial of degree $m \in \mathbb{N}$ if and only if there exist subspaces \mathcal{H}_v and \mathcal{H}_{nil} of \mathcal{H} such that $\mathcal{H} = \mathcal{H}_v \oplus \mathcal{H}_{nil}$ and each T_i admits a representation*

$$T_i = \begin{bmatrix} V_i & * \\ 0 & N_i \end{bmatrix}, \quad i = 1, \dots, n,$$

where $V := [V_1, \dots, V_n] \in [B(\mathcal{H}_v)^n]_1^-$ is a pure row isometry and $N := [N_1, \dots, N_n] \in [B(\mathcal{H}_{nil})^n]_1^-$ is a nilpotent row contraction of order m . Moreover, the degree of Θ_T is the smallest possible order of N in the representation of T .

Proof. Since T is c.n.c. row contraction, we must have $\mathcal{H}_c = \{0\}$. Applying Theorem 1.3, the result follows. \square

We remark that the map $T \mapsto (\Gamma(T), \Theta_T)$ detects the c.n.c. row contractions and completely classify them. Indeed, Corollary 2.6 above and Theorem 5.4 from [5], imply that T and T' are unitarily equivalent c.n.c. row contractions if and only if $\Gamma(T)$ and $\Gamma(T')$ are in $\mathbb{N}_\infty \times \mathbb{N}_\infty \times \{0\}$ and the characteristic functions Θ_T and $\Theta_{T'}$ coincide.

The next result is a characterization of row contractions with constant characteristic function.

Theorem 2.7. *Let $T := [T_1, \dots, T_n] \in [B(\mathcal{H})^n]_1^-$ be a row contraction. Then the following statements are equivalent:*

- (i) the characteristic function Θ_T is a constant, i.e., $\Theta_T = \Theta_T(0)$;
- (ii) $\Gamma(T) \in \mathbb{N}_\infty \times \{0\} \times \mathbb{N}_\infty$;
- (iii) T admits the canonical representation

$$T_i = \begin{bmatrix} V_i & * \\ 0 & W_i \end{bmatrix}, \quad i = 1, \dots, n,$$

where $V := [V_1, \dots, V_n] \in [B(\mathcal{H}_v)^n]_1^-$ is a pure isometry and $W := [W_1, \dots, W_n] \in [B(\mathcal{H}_c)^n]_1^-$ is a coisometry.

If, in addition, T is c.n.u., then Θ_T is constant if and only if T has the representation above where V is a pure isometry and W is a c.n.u. coisometry.

Proof. Using Theorem 1.3, Corollary 1.5, and the definition of the map Γ , the result follows. \square

3. THE AUTOMORPHISM GROUP $\text{Aut}(B(\mathcal{H})_1^n)$ AND UNITARY PROJECTIVE REPRESENTATION

The theory of noncommutative characteristic functions for row contractions [5] was used in [16] to determine the group $\text{Aut}(B(\mathcal{H})_1^n)$ of all free holomorphic automorphisms of the noncommutative ball $[B(\mathcal{H})^n]_1$. We showed that any $\Psi \in \text{Aut}(B(\mathcal{H})_1^n)$ has the form

$$\Psi = \Phi_U \circ \Psi_\lambda,$$

where Φ_U is an automorphism implemented by a unitary operator U on \mathbb{C}^n , i.e.,

$$\Phi_U(X_1, \dots, X_n) := [X_1, \dots, X_n]U, \quad (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1,$$

and Ψ_λ is an involutive free holomorphic automorphism associated with $\lambda := \Psi^{-1}(0) \in \mathbb{B}_n$. The automorphism $\Psi_\lambda : [B(\mathcal{H})^n]_1 \rightarrow [B(\mathcal{H})^n]_1$ is given by

$$\Psi_\lambda(X_1, \dots, X_n) := \lambda - \Delta_\lambda \left(I_{\mathcal{H}} - \sum_{i=1}^n \bar{\lambda}_i X_i \right)^{-1} [X_1, \dots, X_n] \Delta_\lambda^*, \quad (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1,$$

where Δ_λ and Δ_λ^* are the defect operators associated with the row contraction $\lambda := [\lambda_1, \dots, \lambda_n]$. Note that, when $\lambda = 0$, we have $\Psi_0(X) = -X$. We recall that if $\lambda \in \mathbb{B}_n \setminus \{0\}$ and $\gamma := \frac{1}{\|\lambda\|_2}$, then Ψ_λ is a free holomorphic function on $[B(\mathcal{H})^n]_\gamma$ which has the following properties:

- (i) $\Psi_\lambda(0) = \lambda$ and $\Psi_\lambda(\lambda) = 0$;
- (ii) Ψ_λ is an involution, i.e., $\Psi_\lambda(\Psi_\lambda(X)) = X$ for any $X \in [B(\mathcal{H})^n]_\gamma$;
- (iii) Ψ_λ is a free holomorphic automorphism of the noncommutative unit ball $[B(\mathcal{H})^n]_1$;
- (iv) Ψ_λ is a homeomorphism of $[B(\mathcal{H})^n]_1^-$ onto $[B(\mathcal{H})^n]_1^-$.

We say that a row contraction $T = (T_1, \dots, T_n) \in [B(\mathcal{H})^n]_1^-$ is homogeneous if T is unitarily equivalent to $\varphi(T)$ for any $\varphi = (\varphi_1, \dots, \varphi_n) \in \text{Aut}(B(\mathcal{H})_1^n)$.

Theorem 3.1. *Let $T := [T_1, \dots, T_n] \in [B(\mathcal{H})^n]_1^-$ be a completely non-coisometric row contraction. Then T is homogeneous if and only if $\Theta_T \circ \Psi^{-1}$ coincides with the characteristic function Θ_T for any $\Psi \in \text{Aut}(B(\mathcal{H})_1^n)$.*

Proof. Let $\Psi := \Phi_U \circ \Psi_\lambda$ be a free holomorphic automorphism of $[B(\mathcal{H})^n]_1$, where U is a unitary operator on \mathbb{C}^n and $\lambda \in \mathbb{B}_n$. According to [16], the characteristic function has the property that

$$\Theta_{\Psi(T)}(X) = -(I_{\mathcal{G}} \otimes \Omega^*)(\Theta_T \circ \Psi^{-1})(X)(I_{\mathcal{G}} \otimes \Omega_* \mathbf{U}), \quad X \in [B(\mathcal{G})^n]_1,$$

where Ω and Ω_* are the unitary operators. Therefore, $\Theta_T \circ \Psi^{-1}$ coincides with the characteristic function $\Theta_{\Psi(T)}$ for any $\Psi \in \text{Aut}(B(\mathcal{H})_1^n)$. Since T and $\Psi(T)$ are c.n.c. row contractions, we can apply Theorem 5.4 from [5], to deduce that T is homogeneous if and only if Θ_T coincides with $\Theta_{\Psi(T)}$. Consequently, T is homogeneous if and only if Θ_T coincides with $\Theta_T \circ \Psi^{-1}$ for any $\Psi \in \text{Aut}(B(\mathcal{H})_1^n)$. The proof is complete. \square

Lemma 3.2. *Let Φ_k, Φ, Γ_p , and Γ be in the automorphism group $\text{Aut}(B(\mathcal{H})_1^n)$, where $k, p \in \mathbb{N}$. If $\Phi_k \rightarrow \Phi$ and $\Gamma_p \rightarrow \Gamma$ uniformly on $[B(\mathcal{H})^n]_1^-$, then $\Phi_k \circ \Gamma_p \rightarrow \Phi \circ \Gamma$ uniformly on $[B(\mathcal{H})^n]_1^-$, as $k, p \rightarrow \infty$.*

Proof. Since $\Phi \in \text{Aut}(B(\mathcal{H})_1^n)$, it is uniformly continuous on $[B(\mathcal{H})^n]_1^-$. Hence, for any $\epsilon > 0$, there is $\delta > 0$ such that $\|\Phi(Y) - \Phi(Z)\| < \frac{\epsilon}{2}$ for any $Y, Z \in [B(\mathcal{H})^n]_1^-$ with $\|Y - Z\| < \delta$. Taking into account that $\Gamma_p \rightarrow \Gamma$ uniformly on $[B(\mathcal{H})^n]_1^-$, we find $N \in \mathbb{N}$ such that $\|\Gamma_p - \Gamma\|_\infty < \delta$ for any $p \geq N$. Hence, we have

$$\|\Phi(\Gamma_p(X)) - \Phi(\Gamma(X))\| < \frac{\epsilon}{2}$$

for any $X \in [B(\mathcal{H})^n]_1^-$ and $p \geq N$. Consequently, we have

$$\begin{aligned} \|(\Phi_k \circ \Gamma_p)(X) - (\Phi \circ \Gamma)(X)\| &\leq \|(\Phi_k - \Phi)(\Gamma_p(X))\| + \|\Phi(\Gamma_p(X)) - \Phi(\Gamma(X))\| \\ &\leq \|\Phi_k - \Phi\|_\infty + \frac{\epsilon}{2} \end{aligned}$$

for any $X \in [B(\mathcal{H})^n]_1^-$, $k \in \mathbb{N}$, and $p \geq N$. Since $\|\Phi_k - \Phi\|_\infty \rightarrow 0$ as $k \rightarrow \infty$, there is $M \in \mathbb{N}$ such that $\|\Phi_k - \Phi\|_\infty < \frac{\epsilon}{2}$ for any $k \geq M$. Combining these inequalities, we deduce that $\|\Phi_k \circ \Gamma_p - \Phi \circ \Gamma\|_\infty < \epsilon$ for any $p \geq N$ and $k \geq M$, which completes the proof. \square

Let $\phi, \psi \in \text{Aut}(B(\mathcal{H})_1^n)$ and define

$$d_{\mathcal{E}}(\phi, \psi) := \|\phi - \psi\|_\infty + \|\phi^{-1}(0) - \psi^{-1}(0)\|.$$

One can easily check that $d_{\mathcal{E}}$ is a metric on $\text{Aut}(B(\mathcal{H})_1^n)$.

Lemma 3.3. *Let $\Phi_k = \Phi_{U^{(k)}} \circ \Psi_{\lambda^{(k)}}$, $k \in \mathbb{N}$, and $\Phi = \Phi_U \circ \Psi_\lambda$ be free holomorphic automorphisms of the noncommutative ball $[B(\mathcal{H})^n]_1$, where $U^{(k)}, U \in \mathcal{U}(\mathbb{C}^n)$ and $\lambda^{(k)}, \lambda \in \mathbb{B}_n$. Then the following statements are equivalent:*

- (i) $\Phi_k \rightarrow \Phi$ in the metric $d_{\mathcal{E}}$;
- (ii) $U^{(k)} \rightarrow U$ in $B(\mathbb{C}^n)$ and $\lambda^{(k)} \rightarrow \lambda$ in the Euclidean norm of \mathbb{B}_n ;
- (iii) $\Phi_{U^{(k)}} \rightarrow \Phi_U$ and $\Psi_{\lambda^{(k)}} \rightarrow \Psi_\lambda$ uniformly on $[B(\mathcal{H})^n]_1^-$.

Proof. First, we prove that (ii) is equivalent to (iii). Assume that $U^{(k)} = [u_{ij}^{(k)}]_{n \times n}$, $k \in \mathbb{N}$, and $U = [u_{ij}]_{n \times n}$ are unitary matrices with scalar entries, and $\Phi_{U^{(k)}} \rightarrow \Phi_U$ uniformly on $[B(\mathcal{H})^n]_1^-$, as $k \rightarrow \infty$. For each $j = 1, \dots, n$, denote $\mathbf{I}_j := [0, \dots, I, \dots, 0]$, where the identity is on the j -position. Since $\|\Phi_{U^{(k)}}(\mathbf{I}_i) - \Phi_U(\mathbf{I}_i)\| = \left(\sum_{j=1}^n |u_{ij}^{(k)} - u_{ij}|^2\right)^{1/2}$, it is clear that, for each $i, j \in \{1, \dots, n\}$, $u_{ij}^{(k)} \rightarrow u_{ij}$ as $k \rightarrow \infty$. Hence, $U^{(k)} \rightarrow U$ in $B(\mathbb{C}^n)$. Conversely, assume that the latter condition holds. Since $\|\Phi_{U^{(k)}}(X) - \Phi_U(X)\| \leq \|X\| \|U^{(k)} - U\|$ for any $X = [X_1, \dots, X_n] \in [B(\mathcal{H})^n]_1^-$, we deduce that $\Phi_{U^{(k)}} \rightarrow \Phi_U$ uniformly on $[B(\mathcal{H})^n]_1^-$.

Now we prove that $\lambda^{(k)} \rightarrow \lambda$ in the Euclidean norm of \mathbb{B}_n if and only if $\Psi_{\lambda^{(k)}} \rightarrow \Psi_\lambda$ uniformly on $[B(\mathcal{H})^n]_1^-$. Since $\Psi_{\lambda^{(k)}}(0) = \lambda^{(k)}$ and $\Psi_\lambda(0) = \lambda$, one implication is clear. To prove the converse, assume that $\lambda^{(k)} \rightarrow \lambda$ in the Euclidean norm of \mathbb{B}_n . Since the right creation operators R_1, \dots, R_n are isometries with orthogonal ranges, we have

$$\left\| \sum_{i=1}^n \bar{\lambda}_i R_i \right\| = \left\| \left(\sum_{i=1}^n \lambda_i R_i^* \right) \left(\sum_{i=1}^n \bar{\lambda}_i R_i \right) \right\|^{1/2} = \left(\sum_{i=1}^n |\lambda_i|^2 \right)^{1/2} < 1.$$

Consequently, $\left(\sum_{i=1}^n \bar{\lambda}_i^{(k)} R_i \right)^{-1}$ converges to $\left(\sum_{i=1}^n \bar{\lambda}_i R_i \right)^{-1}$, as $k \rightarrow \infty$, in the operator norm. Taking into account that

$$\widehat{\Psi}_\lambda = \lambda - \Delta_\lambda \left(I - \sum_{i=1}^n \bar{\lambda}_i R_i \right)^{-1} [R_1, \dots, R_n] \Delta_{\lambda^*}$$

and a similar relation holds for $\widehat{\Psi}_{\lambda^{(k)}}$, we deduce that $\widehat{\Psi}_{\lambda^{(k)}} \rightarrow \widehat{\Psi}_\lambda$ in the operator norm. Due to the noncommutative von Neumann inequality [6], we have $\|\Psi_{\lambda^{(k)}}(X) - \Psi_\lambda(X)\| \leq \|\widehat{\Psi}_{\lambda^{(k)}} - \widehat{\Psi}_\lambda\|$ for any $X = [X_1, \dots, X_n] \in [B(\mathcal{H})^n]_1^-$. Hence, $\Psi_{\lambda^{(k)}} \rightarrow \Psi_\lambda$ uniformly on $[B(\mathcal{H})^n]_1^-$, which proves our assertion. Therefore, (ii) is equivalent to (iii).

Now, we prove that (i) \implies (ii). Assume that $d_{\mathcal{E}}(\Phi_k, \Phi) \rightarrow 0$ as $k \rightarrow \infty$. Hence, $\Phi_k \rightarrow \Phi$ uniformly on $[B(\mathcal{H})^n]_1^-$ and $\lambda^{(k)} = \Phi_k^{-1}(0) \rightarrow \lambda = \Phi^{-1}(0)$ in \mathbb{B}_n . Consequently, as proved above, we have that $\Psi_{\lambda^{(k)}} \rightarrow \Psi_\lambda$ uniformly on $[B(\mathcal{H})^n]_1^-$. Using Lemma 3.2 and the fact that $\Phi_k = \Phi_{U^{(k)}} \circ \Psi_{\lambda^{(k)}}$, $k \in \mathbb{N}$, and $\Phi = \Phi_U \circ \Psi_\lambda$, we deduce that

$$\Phi_{U^{(k)}} = \Phi_k \circ \Psi_{\lambda^{(k)}} \rightarrow \Phi \circ \Psi_\lambda = \Phi_U$$

uniformly on $[B(\mathcal{H})^n]_1^-$. Hence, $U^{(k)} \rightarrow U$ in $B(\mathbb{C}^n)$ and, therefore, (ii) holds.

It remains to prove that (ii) \implies (i). Assume that (ii) holds. As proved above, $\Phi_{U^{(k)}} \rightarrow \Phi_U$ and $\Psi_{\lambda^{(k)}} \rightarrow \Psi_\lambda$ uniformly on $[B(\mathcal{H})^n]_1^-$. By Lemma 3.2, we deduce that

$$\Phi_k = \Phi_{U^{(k)}} \circ \Psi_{\lambda^{(k)}} \rightarrow \Phi = \Phi_U \circ \Psi_\lambda$$

uniformly on $[B(\mathcal{H})^n]_1^-$. On the other hand, we have $\Phi_k^{-1}(0) = \lambda^{(k)} \rightarrow \lambda = \Phi^{-1}(0)$ in \mathbb{B}_n . Now, one can easily see that $d_{\mathcal{E}}(\Phi_k, \Phi) \rightarrow 0$ as $k \rightarrow \infty$. The proof is complete. \square

After these preliminaries, we can prove the following

Theorem 3.4. *The free holomorphic automorphism group $\text{Aut}(B(\mathcal{H})_1^n)$ is a σ -compact, locally compact topological group with respect to the topology induced by the metric $d_{\mathcal{E}}$.*

Proof. First, we prove that the map

$$\text{Aut}(B(\mathcal{H})_1^n \times \text{Aut}(B(\mathcal{H})_1^n) \ni (\Phi, \Gamma) \mapsto \Phi \circ \Gamma \in \text{Aut}(B(\mathcal{H})_1^n$$

is continuous when $\text{Aut}(B(\mathcal{H})_1^n$ has the topology induced by the metric $d_{\mathcal{E}}$. For $k, p \in \mathbb{N}$, let

$$\begin{aligned} \Phi_k &= \Phi_{U^{(k)}} \circ \Psi_{\lambda^{(k)}}, & \Gamma_p &= \Phi_{W^{(p)}} \circ \Psi_{\mu^{(p)}}, \\ \Phi &= \Phi_U \circ \Psi_{\lambda}, & \Gamma &= \Phi_W \circ \Psi_{\mu}, \end{aligned}$$

be free holomorphic automorphisms of $[B(\mathcal{H})_1^n]_1$, in standard decomposition. Then $U^{(k)}, W^{(p)}, U, W$ are unitary operators on \mathbb{C}^n and $\lambda^{(k)}, \mu^{(p)}, \lambda, \mu$ are in \mathbb{B}_n satisfying relations

$$\lambda^{(k)} = \Phi_k^{-1}(0), \quad \mu^{(p)} = \Gamma_p^{-1}(0), \quad \lambda = \Phi^{-1}(0), \quad \text{and} \quad \mu = \Gamma^{-1}(0).$$

Since $\Phi_k \circ \Gamma_p \in \text{Aut}(B(\mathcal{H})_1^n)$, it has the standard representation

$$(3.1) \quad \Phi_k \circ \Gamma_p = \Phi_{\Omega^{(kp)}} \circ \Psi_{z^{(kp)}}$$

for some unitary operator $\Omega^{(kp)} \in \mathcal{U}(\mathbb{C}^n)$ and $z^{(kp)} \in \mathbb{B}_n$. Note that

$$z^{(kp)} = (\Phi_k \circ \Gamma_p)^{-1}(0) = (\Psi_{\mu^{(p)}}^{-1} \circ \Phi_{W^{(p)}}^{-1} \circ \Phi_k^{-1})(0) = \Psi_{\mu^{(p)}} \left(\lambda^{(k)} W^{(p)*} \right).$$

Similarly, since $\Phi \circ \Gamma \in \text{Aut}(B(\mathcal{H})_1^n)$, we have $\Phi \circ \Gamma = \Phi_{\Omega} \circ \Psi_z$ for some $\Omega \in \mathcal{U}(\mathbb{C}^n)$ and $z = \Psi_{\mu}(\lambda W^*) \in \mathbb{B}_n$. Assume that $d_{\mathcal{E}}(\Phi_k, \Phi) \rightarrow 0$ as $k \rightarrow \infty$ and $d_{\mathcal{E}}(\Gamma_p, \Gamma) \rightarrow 0$ as $p \rightarrow \infty$. According to Lemma 3.3, $\lambda^{(k)} \rightarrow \lambda$ in \mathbb{B}_n and $W^{(p)} \rightarrow W$ in $B(\mathbb{C}^n)$. Hence, $\lambda^{(k)} W^{(p)*} \rightarrow \lambda W^*$ in $B(\mathbb{C}^n)$. Applying again Lemma 3.3, we deduce that $\Psi_{\mu^{(p)}} \rightarrow \Psi_{\mu}$ uniformly on $[B(\mathcal{H})_1^n]_1^-$. Consequently,

$$z^{(kp)} = \Psi_{\mu^{(p)}} \left(\lambda^{(k)} W^{(p)*} \right) \rightarrow z = \Psi_{\mu}(\lambda W^*) \in \mathbb{B}_n$$

as $k, p \rightarrow \infty$. This implies that $\Psi_{z^{(kp)}} \rightarrow \Psi_z$ uniformly on $[B(\mathcal{H})_1^n]_1^-$. On the other hand, since $\Phi_k \rightarrow \Phi$ and $\Gamma_p \rightarrow \Gamma$ uniformly on $[B(\mathcal{H})_1^n]_1^-$, Lemma 3.2 shows that $\Phi_k \circ \Gamma_p \rightarrow \Phi \circ \Gamma$ uniformly on $[B(\mathcal{H})_1^n]_1^-$ as $k, p \rightarrow \infty$. Now, by relation (3.1) and Lemma 3.2, we deduce that

$$\Phi_{\Omega^{(kp)}} = (\Phi_k \circ \Gamma_p) \circ \Psi_{z^{(kp)}} \rightarrow (\Phi \circ \Gamma) \circ \Psi_z = \Phi_{\Omega}$$

uniformly on $[B(\mathcal{H})_1^n]_1^-$. This implies that $\Omega^{(kp)} \rightarrow \Omega$ in $B(\mathbb{C}^n)$ as $k, p \rightarrow \infty$. Using again Lemma 3.3, we conclude that $\Phi_k \circ \Gamma_p \rightarrow \Phi \circ \Gamma$, which proves our assertion.

In what follows, we show that the map $\Phi \mapsto \Phi^{-1}$ is continuous on $\text{Aut}(B(\mathcal{H})_1^n)$ with the topology induced by the metric $d_{\mathcal{E}}$. Assume that $d_{\mathcal{E}}(\Phi_k, \Phi) \rightarrow 0$ as $k \rightarrow \infty$. Using the same notations as above, we have $\Phi_{U^{(k)}} \rightarrow \Phi_U$ and $\Psi_{\lambda^{(k)}} \rightarrow \Psi_{\lambda}$ uniformly on $[B(\mathcal{H})_1^n]_1^-$. Applying Lemma 3.2, we deduce that

$$(3.2) \quad \Phi_k^{-1} = \Psi_{\lambda^{(k)}} \circ \Phi_{U^{(k)}}^* \rightarrow \Psi_{\lambda} \circ \Phi_U^* = \Phi^{-1}$$

uniformly on $[B(\mathcal{H})_1^n]_1^-$, as $k \rightarrow \infty$. On the other hand, we have the standard representations $\Phi_k^{-1} = \Phi_{W^{(k)}} \circ \Psi_{z^{(k)}}$ and $\Phi_k^{-1} = \Phi_W \circ \Psi_z$ for some unitary operators $W^{(k)}, W \in B(\mathbb{C}^n)$ and $z^{(k)}, z \in \mathbb{B}_n$. Note that $z^{(k)} = \Phi_k(0) = (\Phi_{U^{(k)}} \circ \Psi_{\lambda^{(k)}})(0) = \lambda^{(k)} U^{(k)}$ and $z = \Phi(0) = \lambda U$. Since $\lambda^{(k)} \rightarrow \lambda$ in \mathbb{B}_n , we have $z^{(k)} \rightarrow z$ in \mathbb{B}_n , which implies $\Psi_{z^{(k)}} \rightarrow \Psi_z$ uniformly on $[B(\mathcal{H})_1^n]_1^-$, as $k \rightarrow \infty$. Using relation (3.2) and Lemma 3.2, we deduce that

$$\Phi_{W^{(k)}} = \Phi_k^{-1} \circ \Phi_{z^{(k)}} \rightarrow \Phi_W = \Phi^{-1} \circ \Psi_z$$

uniformly on $[B(\mathcal{H})_1^n]_1^-$. Applying Lemma 3.3, we conclude that $\Phi_k^{-1} \rightarrow \Phi^{-1}$ in the topology induced by the metric $d_{\mathcal{E}}$.

Each free holomorphic automorphism $\Phi \in \text{Aut}(B(\mathcal{H})_1^n)$ has a unique representation $\Phi = \Phi_U \circ \Psi_{\lambda}$, where $\lambda := \Phi^{-1}(0)$ and $U \in \mathcal{U}(\mathbb{C}^n)$. This generates a bijection $\chi : \text{Aut}(B(\mathcal{H})_1^n) \rightarrow \mathcal{U}(\mathbb{C}^n) \times \mathbb{B}_n$ by setting $\chi(\Phi) := (U, \lambda)$. According to Lemma 3.3, the map χ is a homeomorphism of topological spaces, where $\text{Aut}(B(\mathcal{H})_1^n)$ has the topology induced by the metric $d_{\mathcal{E}}$ and $\mathcal{U}(\mathbb{C}^n) \times \mathbb{B}_n$ has the natural topology.

Consequently, since $\mathcal{U}(\mathbb{C}^n) \times \mathbb{B}_n$ is a σ -compact, locally compact topological space, so is the automorphism group $\text{Aut}(B(\mathcal{H})_1^n)$. The proof is complete. \square

Corollary 3.5. *The free holomorphic automorphism group $\text{Aut}(B(\mathcal{H})_1^n)$ is path connected.*

Proof. Fix a unitary operator $U \in \mathcal{U}(\mathbb{C}^n)$ and $\lambda \in \mathbb{B}_n$. Since the unitary group $\mathcal{U}(\mathbb{C}^n)$ is path connected, there is a continuous map $[0, 1] \ni t \mapsto U_t \in \mathcal{U}(\mathbb{C}^n)$ such that $U_0 = I$ and $U_1 = U$. Using Lemma 3.3, we deduce that the map $\varphi : [0, 1] \rightarrow \text{Aut}(B(\mathcal{H})_1^n)$ defined by $\varphi(t) := \Phi_{U_t} \circ \Psi_{t\lambda}$ is continuous with respect to the metric $d_{\mathcal{E}}$. Since $\varphi(0) = \Psi_0$ and $\varphi(1) = \Phi_U \circ \Psi_{\lambda}$, the proof is complete. \square

Let $\text{Aut}(B(\mathcal{H})_1^n)$ be the free holomorphic automorphism group of the noncommutative ball $[B(\mathcal{H})_1^n]_1$ and let $\mathcal{U}(\mathcal{K})$ be the unitary group on the Hilbert space \mathcal{K} . According to Theorem 3.4, $\text{Aut}(B(\mathcal{H})_1^n)$ is a topological group with respect to the metric $d_{\mathcal{E}}$. A map $\pi : \text{Aut}(B(\mathcal{H})_1^n) \rightarrow \mathcal{U}(\mathcal{K})$ is called (unitary) projective representation if the following conditions are satisfied:

- (i) $\pi(id) = I$, where id is the identity on $[B(\mathcal{H})_1^n]_1$;
- (ii) $\pi(\varphi)\pi(\psi) = c(\varphi, \psi)\pi(\varphi \circ \psi)$, for any $\varphi, \psi \in \text{Aut}(B(\mathcal{H})_1^n)$, where $c(\varphi, \psi)$ is a complex number with $|c(\varphi, \psi)| = 1$;
- (iii) the map $\text{Aut}(B(\mathcal{H})_1^n) \ni \varphi \mapsto \langle \pi(\varphi)\xi, \eta \rangle \in \mathbb{C}$ is continuous for each $\xi, \eta \in \mathcal{K}$.

Theorem 3.6. *Any completely non-coisometric row contraction $T := [T_1, \dots, T_n] \in [B(\mathcal{H})_1^n]_1^-$ with constant characteristic function is homogeneous. If T is irreducible, then the following statements hold:*

- (i) $\varphi_i(T) = U_{\varphi}^* T_i U_{\varphi}$ for all $\varphi \in \text{Aut}(B(\mathcal{H})_1^n)$, where $U_{\varphi} \in B(F^2(H_n))$ is a unitary operator and

$$U_{\varphi} U_{\psi} = c(\varphi, \psi) U_{\varphi \circ \psi}, \quad \varphi, \psi \in \text{Aut}(B(\mathcal{H})_1^n),$$

for some complex number $c(\varphi, \psi) \in \mathbb{T}$.

- (ii) the map $\varphi \rightarrow U_{\varphi}^*$ is continuous from the uniform topology to the strong operator topology.
- (iii) The map $\pi : \text{Aut}(B(\mathcal{H})_1^n) \rightarrow B(F^2(H_n))$ defined by $\pi(\varphi) := U_{\varphi}$ is a projective representation of the automorphism group $\text{Aut}(B(\mathcal{H})_1^n)$.

Proof. The fact that T is homogeneous follows from Theorem 3.1 using the fact that the characteristic function is constant, i.e., $\Theta_T = \Theta_T(0)$. According to Theorem 2.7, if $T := [T_1, \dots, T_n] \in [B(\mathcal{H})_1^n]_1^-$ is a c.n.c row contraction with polynomial characteristic function, then the characteristic function Θ_T is a constant if and only if T is a pure isometry. Consequently, if T is irreducible we can assume, without loss of generality, that $T = [S_1, \dots, S_n]$.

Let $\varphi = (\varphi_1, \dots, \varphi_n) \in \text{Aut}(B(\mathcal{H})_1^n)$ and let $\widehat{\varphi} = (\widehat{\varphi}_1, \dots, \widehat{\varphi}_n)$ be its model boundary function. Note that $\widehat{\varphi}$ is a pure row isometry and $\widehat{\varphi}_i = \varphi_i(S_1, \dots, S_n)$ for $i = 1, \dots, n$. Using the noncommutative Poisson transform at $\widehat{\varphi}$, we obtain

$$(3.3) \quad \varphi_i(S_1, \dots, S_n) = P_{\widehat{\varphi}}(S_i) = K_{\widehat{\varphi}}^*(S_i \otimes I_{\mathcal{D}_{\widehat{\varphi}}}) K_{\widehat{\varphi}}, \quad i = 1, \dots, n,$$

where the Poisson kernel $K_{\widehat{\varphi}} : F^2(H_n) \rightarrow F^2(H_n) \otimes \mathcal{D}_{\widehat{\varphi}}$ is an isometry. On the other hand, since $\widehat{\varphi}^* \widehat{\varphi} = I$, the characteristic function $\tilde{\Theta}_{\widehat{\varphi}} = 0$. Since $I - \tilde{\Theta}_{\widehat{\varphi}} \tilde{\Theta}_{\widehat{\varphi}}^* = K_{\widehat{\varphi}} K_{\widehat{\varphi}}^*$, we have $K_{\widehat{\varphi}} K_{\widehat{\varphi}}^* = I$, which implies that $K_{\widehat{\varphi}}$ is a unitary operator.

According to [16], $\varphi = \Phi_U \circ \Psi_{\lambda}$, where $\lambda := (\lambda_1, \dots, \lambda_n) = \varphi^{-1}(0) \in \mathbb{B}_n$ and U is unitary operator on \mathbb{C}^n . Moreover, we have

$$\Delta_{\Psi_{\lambda}}^2 = \Delta_{\lambda} \left(I - \sum_{i=1}^n \bar{\lambda} S_i \right)^{-1} P_{\mathbb{C}} \left(I - \sum_{i=1}^n \lambda_i S_i^* \right)^{-1} \Delta_{\lambda}.$$

Therefore, there is a unitary operator $\Lambda_{\lambda} : \mathcal{D}_{\Psi_{\lambda}} \rightarrow \mathbb{C}$ defined by

$$\begin{aligned} \Lambda_{\lambda} \Delta_{\Psi_{\lambda}} f &:= (1 - \|\lambda\|_2^2)^{1/2} P_{\mathbb{C}} \left(I - \sum_{i=1}^n \lambda_i S_i^* \right)^{-1} f \\ &= (1 - \|\lambda\|_2^2)^{1/2} f(\lambda) \end{aligned}$$

for any $f \in F^2(H_n)$. Hence, we deduce that

$$\Lambda_\lambda(z\Delta_{\hat{\Psi}_\lambda}(1)) = z(1 - \|\lambda\|_2^2)^{1/2}, \quad z \in \mathbb{C},$$

and $\mathcal{D}_{\hat{\Psi}_\lambda} = \mathbb{C}\Delta_{\hat{\Psi}_\lambda}(1)$. Since $\Delta_{\hat{\varphi}} = \Delta_{\hat{\Psi}_\lambda}$, we deduce that the operator $W_{\hat{\varphi}} : F^2(\mathcal{H}_n) \otimes \mathcal{D}_{\hat{\varphi}} \rightarrow F^2(H_n)$ defined by

$$W_{\hat{\varphi}}(g \otimes z\Delta_{\hat{\varphi}}(1)) := z(1 - \|\lambda\|_2^2)^{1/2}g, \quad g \in F^2(H_n) \text{ and } z \in \mathbb{C},$$

is unitary. Consequently, we have

$$(3.4) \quad W_{\hat{\varphi}}^*(g) = g \otimes \frac{1}{(1 - \|\lambda\|_2^2)^{1/2}}\Delta_{\hat{\varphi}}(1), \quad g \in F^2(H_n).$$

Setting $U_\varphi := W_{\hat{\varphi}}K_{\hat{\varphi}}$, relation (3.3) implies

$$\varphi_i(S_1, \dots, S_n) = U_\varphi^* S_i U_\varphi, \quad i = 1, \dots, n,$$

for any $\varphi \in \text{Aut}(B(\mathcal{H})_1^n)$. Hence, if $\psi \in \text{Aut}(B(\mathcal{H})_1^n)$, then

$$(3.5) \quad (\varphi_i \circ \psi)(S_1, \dots, S_n) = U_{\varphi \circ \psi}^* S_i U_{\varphi \circ \psi}, \quad i = 1, \dots, n.$$

On the other hand, due to Theorem 3.1 from [16], the noncommutative Poisson transform satisfies the relation $P_{\widehat{\varphi \circ \psi}}[\chi] = P_{\widehat{\psi}}P_{\widehat{\varphi}}[\chi]$ for any $\chi \in C^*(S_1, \dots, S_n)$, the Cuntz-Toeplitz C^* -algebra generated by the left creation operators S_1, \dots, S_n . In particular, when $\chi = S_i$, we obtain

$$K_{\widehat{\varphi \circ \psi}}^*(S_i \otimes I_{\widehat{\mathcal{D}_{\varphi \circ \psi}}})K_{\widehat{\varphi \circ \psi}} = K_{\widehat{\psi}}^* \left\{ [K_{\widehat{\varphi}}^*(S_i \otimes I_{\widehat{\mathcal{D}_{\varphi}}})K_{\widehat{\varphi}}] \otimes I_{\widehat{\mathcal{D}_{\psi}}} \right\} K_{\widehat{\psi}}, \quad i = 1, \dots, n.$$

Hence, we deduce that

$$(\varphi_i \circ \psi)(S_1, \dots, S_n) = U_{\widehat{\psi}}^* U_{\varphi}^* S_i U_{\varphi} U_{\widehat{\psi}}, \quad i = 1, \dots, n.$$

Combining this relation with (3.5), we deduce that

$$U_{\varphi \circ \psi}^* S_i U_{\varphi \circ \psi} = U_{\widehat{\psi}}^* U_{\varphi}^* S_i U_{\varphi} U_{\widehat{\psi}}, \quad i = 1, \dots, n,$$

which is equivalent to

$$U_{\varphi} U_{\widehat{\psi}} U_{\varphi \circ \psi}^* S_i = S_i U_{\varphi} U_{\widehat{\psi}} U_{\varphi \circ \psi}^*, \quad i = 1, \dots, n.$$

Since S_1, \dots, S_n is irreducible and $U_{\varphi} U_{\widehat{\psi}} U_{\varphi \circ \psi}^*$ is a unitary operator, we have $U_{\varphi} U_{\widehat{\psi}} U_{\varphi \circ \psi}^* = \overline{c(\varphi, \psi)} I$ for some complex number with $|c(\varphi, \psi)| = 1$. Hence, we deduce that $U_{\varphi} U_{\widehat{\psi}} = c(\varphi, \psi) U_{\varphi \circ \psi}$ for any $\varphi, \psi \in \text{Aut}(B(\mathcal{H})_1^n)$.

Now, we prove part (ii). Let $\varphi^{(p)} := (\varphi_1^{(p)}, \dots, \varphi_n^{(p)})$, $p = 1, 2, \dots$, and $\psi := (\psi_1, \dots, \psi_n)$ be in $\text{Aut}(B(\mathcal{H})_1^n)$ such that $\varphi_i^{(p)}$ converges to ψ_i in the uniform norm on $[B(\mathcal{H})^n]_1$, that is,

$$\|\varphi_i^{(p)} - \psi_i\|_\infty := \sup_{X \in [B(\mathcal{H})^n]_1} \|\varphi_i^{(p)}(X) - \psi_i(X)\| \rightarrow 0 \quad \text{as } p \rightarrow \infty,$$

for $i = 1, \dots, n$. Since $\varphi_i^{(p)}$ and ψ_i are uniformly continuous on $[B(\mathcal{H})^n]_1$, the model boundary functions $\widehat{\varphi_i^{(p)}}$ and $\widehat{\psi_i}$ are in the noncommutative disc algebra \mathcal{A}_n and we have $\widehat{\varphi_i^{(p)}} = \varphi_i^{(p)}(S_1, \dots, S_n)$ and $\widehat{\psi_i} = \psi_i(S_1, \dots, S_n)$. Consequently, the convergence above implies that $\widehat{\varphi_i^{(p)}} \rightarrow \widehat{\psi_i}$ in the operator norm topology. Each $\varphi^{(p)} \in \text{Aut}(B(\mathcal{H})_1^n)$ has the form $\varphi^{(p)} = \Phi_{U_p} \circ \Psi_{\lambda^{(p)}}$, where Φ_{U_p} is an automorphism implemented by a unitary operator U_p on \mathbb{C}^n , i.e.,

$$\Phi_{U_p}(X_1, \dots, X_n) := [X_1, \dots, X_n]U_p, \quad (X_1, \dots, X_n) \in [B(\mathcal{H})^n]_1,$$

and $\Psi_{\lambda^{(p)}}$ is the involutive free holomorphic automorphism associated with $\lambda^{(p)} := (\varphi^{(p)})^{-1}(0) \in \mathbb{B}_n$. Similarly, we have $\psi = \Phi_U \circ \Psi_\mu$, where $U \in B(\mathbb{C}^n)$ is a unitary operator and Ψ_μ is the involutive free holomorphic automorphism associated with $\mu := \psi^{-1}(0) \in \mathbb{B}_n$. Due to the above-mentioned convergences, we deduce that $\varphi^{(p)}(0) \rightarrow \psi(0)$ as $p \rightarrow \infty$. Taking into account that $\Psi_{\lambda^{(p)}}(0) = \lambda^{(p)}$ and $\Psi_\mu(0) = \mu$, we have $\varphi^{(p)}(0) = \lambda^{(p)}U_p$ and $\psi(0) = \mu U$. Therefore, $\lambda^{(p)}U_p$ converges to μU in the operator norm. Since U_p and U are unitary operators, we deduce that $\|\lambda^{(p)}\|_2 \rightarrow \|\mu\|_2$ as $p \rightarrow \infty$.

Given $\epsilon > 0$ and $x = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha e_\alpha \in F^2(H_n)$, let $k \in \mathbb{N}$ be such that $\|x - \sum_{\alpha \in \mathbb{F}_n^+, |\alpha| \leq k} a_\alpha e_\alpha\| < \frac{\epsilon}{4}$. Using relation (3.4) and the properties of the noncommutative Poisson kernel, we have

$$\begin{aligned} \sum_{|\alpha| \leq k} a_\alpha U_{\varphi^{(p)}}^* e_\alpha &= \sum_{|\alpha| \leq k} a_\alpha K_{\widehat{\varphi^{(p)}}}^* W_{\widehat{\varphi^{(p)}}}^* e_\alpha \\ &= \sum_{|\alpha| \leq k} a_\alpha K_{\widehat{\varphi^{(p)}}}^* \left(e_\alpha \otimes \frac{1}{(1 - \|\lambda^{(p)}\|_2^2)^{1/2}} \Delta_{\widehat{\varphi^{(p)}}}(1) \right) \\ &= \sum_{|\alpha| \leq k} a_\alpha \frac{1}{(1 - \|\lambda^{(p)}\|_2^2)^{1/2}} [\widehat{\varphi^{(p)}}]_\alpha \Delta_{\widehat{\varphi^{(p)}}}^2(1). \end{aligned}$$

A similar relation holds if we replace $\varphi^{(p)}$ with ψ . Since $\widehat{\varphi_i^{(p)}} \rightarrow \widehat{\psi_i}$ in the operator norm topology and $\|\lambda^{(p)}\|_2 \rightarrow \|\mu\|_2$ as $p \rightarrow \infty$, there is $N \in \mathbb{N}$ such that

$$\left\| \sum_{|\alpha| \leq k} a_\alpha U_{\varphi^{(p)}}^* e_\alpha - \sum_{|\alpha| \leq k} a_\alpha U_\psi^* e_\alpha \right\| < \frac{\epsilon}{2}$$

for all $p \geq N$. Using the fact that $U_{\varphi^{(p)}}$ and U_ψ are unitary operators, we deduce that

$$\begin{aligned} \|U_{\varphi^{(p)}}^* x - U_\psi^* x\| &\leq \left\| U_{\varphi^{(p)}}^* \left(x - \sum_{|\alpha| \leq k} a_\alpha e_\alpha \right) \right\| + \left\| U_{\varphi^{(p)}}^* \left(\sum_{|\alpha| \leq k} a_\alpha e_\alpha \right) - U_\psi^* \left(\sum_{|\alpha| \leq k} a_\alpha e_\alpha \right) \right\| \\ &\quad + \left\| U_\psi^* \left(x - \sum_{|\alpha| \leq k} a_\alpha e_\alpha \right) \right\| \\ &\leq 2 \left\| x - \sum_{|\alpha| \leq k} a_\alpha e_\alpha \right\| + \left\| U_{\varphi^{(p)}}^* \left(\sum_{|\alpha| \leq k} a_\alpha e_\alpha \right) - U_\psi^* \left(\sum_{|\alpha| \leq k} a_\alpha e_\alpha \right) \right\| \\ &\leq 2 \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for any $p \geq N$. Therefore $U_{\varphi^{(p)}}^*$ converges to U_ψ^* , as $p \rightarrow \infty$, in the strong operator topology.

To prove part (iii), let $\varphi^{(p)}, \varphi$ be in $\text{Aut}(B(\mathcal{H})_1^n)$ be such that $\varphi^{(p)} \rightarrow \varphi$ in the metric $d_{\mathcal{E}}$, as $p \rightarrow \infty$. Then $\|\varphi^{(p)} - \varphi\|_\infty \rightarrow 0$, as $p \rightarrow \infty$. Using (i) and (ii), we deduce that the map $\pi : \text{Aut}(B(\mathcal{H})_1^n) \rightarrow B(F^2(H_n))$ defined by $\pi(\varphi) := U_\varphi$ is a projective representation of the automorphism group $\text{Aut}(B(\mathcal{H})_1^n)$. The proof is complete. \square

We say that two projective representations π_1, π_2 of $\text{Aut}(B(\mathcal{H})_1^n)$ on the Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 , respectively, are equivalent if there exists a unitary operator $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ and a Borel function $\sigma : \text{Aut}(B(\mathcal{H})_1^n) \rightarrow \mathbb{T}$ such that $\pi_2(\varphi)U = \sigma(\varphi)U\pi_1(\varphi)$ for all $\varphi \in \text{Aut}(B(\mathcal{H})_1^n)$.

We remark that if π_1 and π_2 are two projective representations of $\text{Aut}(B(\mathcal{H})_1^n)$ associated with T , as in Theorem 3.6, then we have $\varphi_i(T) = \pi_1(\varphi)^* T_i \pi_1(\varphi)$ and $\varphi_i(T) = \pi_2(\varphi)^* T_i \pi_2(\varphi)$ for all $\varphi \in \text{Aut}(B(\mathcal{H})_1^n)$ and $i = 1, \dots, n$. Hence, $\pi_1(\varphi)\pi_2(\varphi)^*$ commutes with each operator T_1, \dots, T_n . Since $[T_1, \dots, T_n]$ is irreducible, we deduce that $\pi_1(\varphi)\pi_2(\varphi)^* = d(\varphi)I$ for some constant $d(\varphi) \in \mathbb{T}$ which proves that π_1 and π_2 are equivalent.

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